The Mathematics Teacher

FEBRUARY 1961

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A report of a National Science Foundation summer institute in mathematics for high school students at Columbia

GEORGE GROSSMAN

The development of a concept: a demonstration lesson

ROBERT JACKSON

The Mathematics Teacher ***is the official journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges, and Teacher Education Colleges.***

All editorial correspondence, including books for review, should be addressed to the Editor.

All other correspondence should be addressed to

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W., Washington 6, D.C.

Printed at Menasha, Wisconsin, U.S.A. Second-class postage has been paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412, P. L. & R., authorized March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume LIV, *number* 2 February 1961

THE MATHEMATICS TEACHER is published monthly eight times a year, October through May. The individual subscription price of $5.00 ($1.50 to students) includes membership in the Council. For an additional $3.00 ($1.00 to students) the member may also receive *The Arithmetic Teacher.* Institutional subscription to either journal: $7.00 per year. Single copies: 85 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics,* 1201 Sixteenth Street, N.W., Washington 6, D.C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

Rigor vs. intuition in mathematics[*]

JOHN G. KEMENY, *Dartmouth College, Hanover, New Hampshire.*

Three important emphases of mathematics teaching, intuition, abstraction, and creation, are discussed here.

I THOUGHT THAT I might make some informal remarks this evening about the role of rigor and the role of intuition in both mathematical research and in the teaching of mathematics. The best way to contrast these two very important trends is by a story told about one of our leading graduate schools. There was an advanced seminar in topology in which the lecturer devoted the entire hour to writing out a proof with complete rigor. After having filled all the blackboards, he had everyone in the room completely lost, including one of his own colleagues, who jumped up and said, "Look, I just don't understand this proof at all. I tried to follow you, but I got lost somewhere. I just didn't get it at all." The lecturer stopped for a moment, looked at him, and said, "Oh, didn't you see it? You see, it's just that the two spaces connect like this," intertwining his two arms in a picturesque fashion. And then his colleague exclaimed, "Oh, now I get the whole proof."

There is something in this story that is typical of a great deal of mathematical research. You can write down long formulas to make a proof complete and rigorous. Indeed, you *have* to write down long formulas and justify every step. Yet very often there is one key idea which, once understood, makes the rest of it purely routine. And if this one idea is not understood, the whole proof is meaningless to the student or to the research worker.

My basic theme this evening is that I am somewhat worried that amongst all the very fine reforms that are being suggested, tried out, and patiently worked out on the high school level, there may have come an overemphasis on rigor and a playing down of intuition in mathematics. This evening I am going to try to plug for continued emphasis on the role of intuition in the teaching of mathematics.

If you look at the problem historically, you will find that rigor always enters mathematics quite late. Euclidean geometry is identified as one of the high points in mathematical rigor. Yet Euclid is full of holes from any modern standpoint. A great deal of publicity has been given to this, and I won't dwell on it. Let us take only one small point. It is impossible to prove from Euclid's axioms that a circle has an inside and an outside. Of course it is quite important to know that a circle has an inside and an outside; on the other hand, since this proof requires a considerable effort in an advanced graduate course in mathematics, it was perhaps fortunate that Euclid didn't realize that he couldn't prove this. In the last analysis he knew very well that a circle does have an inside and does have an outside. This does not mean that the Jordan Curve Theorem is not of importance to modern mathematics. It is tremendously important, but the fact that this major hole exists in Euclidean geometry does not destroy the value of Euclid's work.

Let me jump over several centuries to Newton's work on the calculus. If any of our freshman students in calculus should perform in the sloppy, unjustifiable way in which Newton did the calculus, we would surely refuse to pass him. Neverthe-

[*] Banquet address, NCTM meeting, Salt Lake City, August 23, 1960.

less, I would like to maintain that Newton's work was, in the long run, quite valuable.

I come to one of the greatest names in the history of mathematics, that of Euler. Euler was quite capable of doing rigorous mathematics, but occasionally he did hair-raising things. His manipulation of some infinite series, for example, was completely unjustifiable. As a matter of fact, it took the next two hundred years of mathematical progress to find out that practically everything that Euler did could be justified, though there was no real reason why Euler should have suspected this. Euler was just "plain lucky"—with dozens of major ideas, almost every one of which turned out to be right. His remarkable mathematical intuition has never been equaled.

The following story is told about a very famous modern mathematician, one of the cofounders of a great branch of mathematics. He had published a certain paper in which he mentioned a theorem without proof, and a Russian mathematician wrote to him, asking whether it would be possible to receive a proof. Our distinguished mathematician answered the Russian request. After about a month or so he had a reply. The Russian mathematician thanked him profusely; however, he had to point out that the proof sent was for a completely different theorem and that the proof was incorrect. As a matter of fact, this particular mathematician is credited with many incorrect proofs, and yet there isn't a single creative mathematician who would not list him as one of the greatest mathematicians of the century.

After this very elaborate historical introduction, I want to ask you a simple question. If it is possible for Euclid, for Newton, for Euler, and for many contemporary mathematicians to go down in history as among the great, even though they were far from completely rigorous, don't you think that the same sin might be forgivable on the part of high school students?

Let me be more specific. Let us talk a little about definitions. Everybody knows that one has to be very rigorous in definitions. There has been a strong trend in recent years to make modern concepts—like that of a function—exceedingly rigorous. I was particularly pleased that in this trend, set theory has played a major role. It so happens that I wrote my Ph.D. thesis on set theory, and therefore I am exceedingly fond of it. Perhaps I may perform a public service by seeing to it that, as long as we define functions in terms of sets, we make sure that the definition is really rigorous.

You all know that a function, when correctly defined, is a set of ordered pairs. Of course, people don't quite take the trouble to distinguish between an ordered pair and an unordered pair. There happens to be a very nice way of doing this. It is the standard method in set theory: to get an ordered pair in set theory (which deals with unordered sets), you just define an ordered pair as a set having two sets as elements. For example, if you want the ordered pair $\langle A, B\rangle$, one of the elements is the pair $\{A, B\}$ and the other element is a set whose only element is A. Now you can identify which is which, because A belongs to both of these and B only to one of them. Then you've got something that will serve as an ordered pair.

The mathematical logician would then define a function as follows: "A function is a set of sets of sets, such that each element is a two-element set consisting of a one-element set and a two-element set, and the one-element set is a subset of the two-element set. Furthermore, for each object whose unit set is an element of the function, there is at most one other object which is an element of one of the elements of the elements of the function to which the given object belongs." I am delighted that this kind of highly enlightening and intuitive material has finally reached the high school level. We wouldn't dream of teaching it to undergraduates in college, but I am delighted to see that you are

teaching this material in high school.

My point obviously is that there *must* be a more intuitive way of defining a function. This is a particularly amusing example to a logician who happens to know the history of this particular subject. Around the turn of the century, for the first time, mathematicians interested in the foundations of mathematics developed two fundamentally different approaches to the foundations of mathematics. In one of them, pioneered by Zermelo, the basic idea was that of *set*. In the other, pioneered by Russell and Whitehead, the basic idea was that of *function*. Of course, both of these concepts are fundamental to mathematics and, therefore, in the theory of types, in which function is the basic idea, they had to introduce sets by some sort of trick. The other basic approach—of set theory—had to introduce functions by a trick. It happened to be a rather complicated trick, but was useful, nevertheless. And therefore, if you insist on making sets fundamental, you have to do something very complicated to define functions. It never occurred to mathematical logicians that anyone would ever use this as the basic definition of a function in an elementary course. Somehow, half of this history has been made public; which shows that certain branches of mathematics ought, perhaps, to be classified.

The impression has been created that the *only* way to define a function is in terms of sets. I'll put it to you that research mathematicians never think of a function in this particular manner. They may differ in ways of looking at it—there are three or four different ways—but they have certain similarities, and they are highly intuitive. A function is a mapping, an assignment where you have certain objects in one set, and to each object you assign some specific object, usually from a different set. Now this is a very simple idea, and of a type met in everyday life. Examples familiar to all students can be given.

For example, every human being has a father and a mother. Well, *father* and *mother* are perfectly good functions, defined over the set of human beings, where to each human being "father of" assigns a specific male human being and "mother of" determines a specific female human being. One can illustrate a great many basic ideas connected with functions in terms of simple examples of this sort.

Let me take one of the ideas that has caused most trouble in the study of functions, namely that of the idea of a composite function, where one function is applied to another function. A student may have trouble at first recognizing as a composite function something like $(3x+2)^4$, but in everyday life it is not terribly hard to explain to him what a maternal grandfather is—father of your mother—and this is a typical composite function.

Actually these simple everyday examples have great pedagogical advantages, because the usual examples of numerical functions are too special for the use of the research mathematician. In advanced mathematics courses you rarely deal with something as simple as a numerical function, and family relations illustrate the general nature of a function much better than do the ordinary numerical functions.

Let us next discuss the simultaneous solution of linear equations (like $3x+4y=5$). There is a very elaborate theory in advanced mathematics, known as linear algebra, which has relevance to some of the material being suggested for high school curricula. But if we attempted a complete, rigorous treatment of everything one ought to know about such equations, it would ruin high school algebra, and I don't believe that anyone has ever suggested this.

But what is commonly taught about these equations? It is usually taught as a bag of tricks and techniques which have two serious limitations. One limitation is that these tricks only work in special situations. For example, one of the favorite

methods of solving equations—by means of determinants—works only if you happen to have the same number of equations as unknowns and where the key determinant is not zero. So for the advanced mathematician, the solution of an equation by determinants is very rarely of value. (Also, the method as usually taught happens to work only for 3×3 determinants.) The second serious objection is that the methods usually taught for solving simultaneous equations are highly dated and have long lost their practical importance. In the age of computing machines, we must rethink what is a practical way of solving equations and what is an impractical way.

On the related topic of finding roots of an algebraic equation, we had a debate once at Dartmouth about how useful Horner's Method is. I offered to put this to a test. A colleague of mine and I each had a desk computer and a fifth-degree equation for which we had to find a root to five decimal places. He was going to use Horner's Method and I was using successive approximations, i.e., organized common sense. I am sorry to say that this particular test of the practicability of Horner's Method turned out to be quite inconclusive because, unfortunately, I had the root to five decimal places before my colleague remembered Horner's Method. Which was a great pity, because it was the first (and presumably last) time in his career as a professional mathematician that he ever had the least excuse for using Horner's Method.

Let me come back to linear equations. What are some of the key ideas? I think there are two key ideas that one should understand about linear equations, and they do not require tremendous rigor, only a feeling for the subject matter and a degree of understanding. First, one must know what it means to solve simultaneous linear equations, and more generally, what it means to solve equations. A great deal of worthwhile work has been done and various different approaches to this have been suggested. Students should understand that solving an equation will mean finding a certain set of numbers about which a particular assertion is true—about which the equation holds. And solving simultaneous equations means finding a set of numbers about which several assertions hold, in other words, numbers which have all these special properties; in short, finding where the intersection of several sets falls.

Secondly, it is important to connect this idea of a set of solutions with geometry, for one's geometric intuition is usually stronger than one's numerical intuition. For example, if you plot the solution of an equation in two unknowns in the plane, it comes out to be a straight line. If we plot a second equation, it is also represented by a straight line; and to ask for numbers (or rather number pairs) which satisfy both of these equations will obviously be asking for the point or points that the two lines have in common; it is the intersection of the two lines. It is very interesting that when you formulate the same thing set-theoretically, there too the word "intersection" occurs.

Let us illustrate in three dimensions the major theorem that all students should know about simultaneous equations. *One of three things must happen: you may have no solutions; you may have a unique solution; or you may have infinitely many solutions.* This is by far the most important fact known about simultaneous linear equations. There are no solutions, one solution, or infinitely many.

If you think of it geometrically, this fact is obvious. In three dimensions each equation represents a plane. Let us begin with two of them, and let us suppose that they intersect in a line L. Now we ask what happens if we add an additional plane. If we have bad luck, the new plane may be parallel to the line L, and we will have no solution. Normally the new plane will cut L in one and only one point, and we have a unique solution. Or it may happen that the third plane passes through L. In

that case we will have infinitely many solutions. This is the most general possible situation for simultaneous linear equations.

It is also easy to see that what happens is not determined by the number of different equations or the number of different unknowns. You could have just two equations in three unknowns and have no solution, because the two planes may be parallel. You may have a hundred different equations in three unknowns and they may still have a unique solution if, by chance, they all go through the same point. And you may have a hundred different equations in three unknowns and you may have infinitely many solutions. Just think of a number of planes, at different angles, all going through a given line.

This simple fact, that I explained here in about five minutes, can be explained to students in one class period. And yet, it is the fundamental fact about solving simultaneous linear equations. It is obvious, *if you teach it intuitively.*

I feel very strongly that, although a degree of rigor is important in teaching because a student should be able to understand what a proof is, it is vastly more important to emphasize basic ideas and to build up the intuition possessed by the student.

Of course, we do not know what constitutes intuition. Even what is intuitively obvious can be a matter of great controversy. You know that the mathematician's favorite word is "trivial," which is a shorthand way of saying "intuitively obvious." There are endless stories about the word "trivial." My favorite is the one about the mathematician who, in a lecture, asserted that a result is trivial. One of his colleagues challenged him, and they got into a long argument which was still going on at the end of the class. The class tiptoed out, and the two mathematicians were seen arguing vehemently for over two hours. When they finally showed up outside, students eagerly queried the challenger about the outcome. He replied: "Oh, he was right. It *is* trivial."

While I maintain that *rigor* is not a necessity in much high school mathematics teaching, I feel quite differently about *abstraction,* which has been tied to it (somewhat accidentally) in many developments. Mathematics by its very nature is abstract. It is the power of abstraction that enables mankind to rise above lower animals. The power to abstract should be developed in students as early as possible.

There has been a feeling that the only way to teach abstraction is to take an abstract axiom system and develop it in detail. This is a worthwhile undertaking; I am not criticizing it. But this is not the only way to develop a feeling for abstraction.

Abstraction should start from simple, concrete examples. An idea can be abstract and still be highly intuitive. For example, a measure space is a very important abstract concept. It is an advanced idea, but it can be explained in the simplest possible terms. (I personally like to do it in terms of probabilities, though it can be done in other ways.)

Just take a collection of objects, say a set of five objects, and assign a weight to each one. Think of each subset as being weighed, literally, by putting all the weights in the subset on scales. This can be used to introduce the basic idea of a measure space. If the weights are all positive and happen to add up to one, you have begun to do probability theory. If the weights don't necessarily add up to one, you may be measuring areas; and if we allow negative weights, then you are doing generalized measure theory—which you are not supposed to be doing until graduate school. It is a natural generalization then to go over to an abstract approach to the idea of area, and a student can see that some of the basic rules governing area govern much wider ranges of mathematical and applied disciplines. Indeed, eventually these same rules are go-

ing to apply to all kinds of integrals found in geometry, in physics, in applications, and in several advanced branches of mathematics.

A second important role of abstraction is to connect unrelated ideas. If you have a large number of unrelated ideas, you have to get quite a distance away from them to be able to get a view of all of them, and this is the role of abstraction. If you look at each one too closely, you see too many details. You have to go far away to see what they have in common. And it is by no means true that if you get far away things are going to become less clear. They may appear simpler, because you can only see the large, broad outlines; you do not get lost in petty details. This has been the secret of a great deal of modern mathematics.

Let me take up two unorthodox examples, one from algebra and one from geometry. In algebra, let us select the idea of an *isomorphism*, which is central in modern algebra. Two structures which are alike are said to be isomorphic. More precisely, you've got two sets and you do something with each of them. If you can match up the objects in the two sets in such a way that whenever you do something in one, exactly the same thing happens to corresponding elements in the other set, then one speaks of an isomorphism. This concept is useful because it is so general.

Let us apply it to something that is well known to you; let us apply it to real numbers and to the operations of multiplication and addition. Take two sets: one consists of the positive real numbers, and the other collection has all the real numbers in it. I'm going to concentrate on one type of operation for each of them. In the first set it will be multiplication; in the second set it will be addition. Let us ask whether we can establish an isomorphism. Is it possible to match up the positive real numbers with all the real numbers in such a way that every time you multiply two positive numbers, and you add the corresponding real numbers, the results will correspond? The answer is "yes," even if we require a "continuous" matching.

As a matter of fact, if you try doing this, you quickly convince yourself that you have a certain amount of freedom. If I use the letter f to stand for the correspondence, and I take the positive real number one, we first find that what will correspond to it will be zero (because if you think of your basic laws of multiplication, one plays exactly the same role for multiplication as zero plays for addition), i.e., $f(1)=0$. If you try raising positive numbers to powers, you will quickly find that this operation corresponds to multiplication, i.e., $f(x^y)=y\cdot f(x)$. After all, raising to powers is essentially repeated multiplication, and hence should correspond to repeated addition, which is multiplication. This formula almost gives us our complete matching, namely, all we have to do is to find a number b, such that $f(b)=1$. The moment you have found that number, b^y will correspond to $y\cdot f(b)$, which is y, and then you have found the whole secret of how to match the two sets. What you have, of course, are logarithms to the base b.

The amount of choice you have is the freedom of choosing a base for logarithms. You quickly find that your base can be anything except one, and therefore you will have as many different isomorphisms here as you can choose bases for logarithms —any positive real number other than one.

If you look at the same mapping in the opposite direction, you have exponential functions. This is one of many useful ways of looking at logarithms and exponentials; what they really do is to establish an isomorphism, a complete structural matching-up, of the positive real numbers under multiplication with real numbers under addition.

I am going to select my other example from topology, the celebrated abstract version of geometry. Let me quote a famous result and show you something that

can be done with it. Take any simple polyhedron, and count the number of vertices, the number of faces, and the number of edges. For example, in a box you find eight corners, so the number of vertices is equal to eight. The number of faces on a box is equal to six. And the number of edges—there are four on top, there are four on the bottom, and there are four on the sides—so there are twelve edges. The number of vertices, plus the number of faces, is fourteen; if you subtract the number of edges, you get two. A remarkable fact, discovered by Euler, is that you can take absolutely any simple polyhedron—any three-dimensional figure with straight edges and plane faces, without holes—and you will always get two as an answer. For example, for a tetrahedron, you get four vertices plus four faces, which equals eight; subtract six edges, and you get two. You can reshape the figure as you like, except that you must not cut a hole in it, because this is a topological property. Actually, there is a more general formula where the number of holes enters into the formula. There are also formulas for other numbers of dimensions.

This fascinating topic should, even without proofs, interest a great many high school students.

But how can we tie this abstract idea to high school topics? Well, for example, one rather isolated, interesting topic in solid geometry (or what is left of solid geometry) is the study of the five regular polyhedra. But why are there just five of them? We will use Euler's formula to answer the question.

Let there be f faces, each being a regular polygon of s sides, and let k faces meet at each vertex. Then the number of edges is $sf/2$ and the number of vertices is sf/k. Hence Euler's formula asserts that $f+sf/k-sf/2=2$ or

$$(2k+2s-ks)f=4k.$$

Obviously, k and s must be at least 3. Thus there are only five possible combinations of integer value for k and s, since larger values would make the left side zero or negative:

$k=3,\ s=3,\ f=4$ (tetrahedron);

$k=3,\ s=4,\ f=6$ (cube);

$k=3,\ s=5,\ f=12$ (dodecahedron);

$k=4,\ s=3,\ f=8$ (octahedron);

$k=5,\ s=3,\ f=20$ (icosahedron).

This simple mixture of intuition and rigorous proof shows us why there are just five regular solids.

Whatever you may think about my views on rigor, intuition, and abstraction, I hope that we have one common goal, to develop in students early their ability to create new ideas. I feel that able students need only a slight lead, especially if you are fortunate enough to have sections in which you separate off the good students. Once they are amongst their peers they can be encouraged to develop ideas freely; though a certain amount of guidance is very important even here.

I recently finished teaching at a summer institute that Dartmouth cosponsored. In addition to a number of high school teachers we had two dozen very able secondary-school juniors. It was an interesting and enlightening experience. The students were wonderful; I would be happy to have had any one of them at Dartmouth. It was an outstandingly able group, but they were badly in need of some channeling of their unguided mathematical abilities. They seemed to be under the impression that the highest possible use of high school mathematics is solving puzzles. This is not too surprising since high school libraries usually have mathematics sections loaded with puzzle books. The rest of their time was spent on problems of the same sort as they had for homework, only harder—something that took more time or ingenuity. Of course, this has some value, but a really good student should be given a task that is somewhat higher and more challenging to him than doing hard versions of homework problems or solving uninstructive puzzles.

The first thing that we can all do is to give the students a good book to read. An able high school student is old enough to read books. There are many books on the market that can be given to a student. SMSG is undertaking a major effort to turn out special pamphlets and monographs for just this type of use in high school, and other groups are doing the same. I saw a list compiled recently (by Mu Alpha Theta, Box 1155, University of Oklahoma, Norman, Oklahoma) that tried to prove to you that for $180 you can build up a superb high school mathematics collection. And, indeed, that list, with $180 worth of books in it is more than enough for any high school collection. Half of it would do very well. I was pleasantly surprised by the number and quality of really good books that are available and readable by high school students.

Even better than giving them books is to give them *good* problems. There are lots of good problems suitable for students. Take a book like that marvelous one by Professor G. Pólya, *Induction and Analogy in Mathematics* (Princeton University Press, 1954). I will cite just one of his classic examples. He asks the question, "Suppose you have a number of planes, say ten planes, and you partition a room with these; into how many pieces will a room be cut with ten partitions? How about n partitions?" I assure you that it is not an easy problem, unless you go at it just right. The right way to solve the three-dimensional problem is to consider it first in one and two dimensions. In one dimension we have a line with n points on it, and the number of pieces is clearly $n+1$. Then do it in two dimensions, using the solution of the one-dimensional case, and then go to three dimensions. Your best student will not only have no trouble in doing this, but will come back with the solution for four, five, and—hopefully—for n dimensions.

Or give these students a little bit of number theory—the theory of whole numbers. There are dozens of opportunities for the student to develop his own formulas. Nothing is more thrilling than to find a mathematical formula that holds without exception. If you want them to do geometry, why stick to three-dimensional geometry? It is rather dull. Why not let them do four- or five-dimensional geometry? Let them try to argue by analogy (or rigorously, if they can) theorems in four- and five-dimensional geometry, using their experience in two and three dimensions. If you want to do abstract algebra, the beginnings of the theory of groups are easily accessible to a high school group. I have often used that as a talk for high school students. The vast majority of them grasped the idea the first time. I certainly don't say that you should do this in a single lecture, but spend a week on it, then challenge them to go out and develop their own examples of groups. Finally—if they are really ambitious—let them form their own axiom system for something more abstract, let them work it out, and see for themselves what actually happens.

I think in all fields we owe it to our best students to encourage creative endeavor. The great advantage that we have in mathematics is that, again and again, examples have shown it possible to get students at a remarkably early age to do creative mathematics. There have been major contributions to mathematics by men in their late teens. Even if your students aren't going to do creative work, at least give them a first taste of developing something that may not be new to the mathematical world, but is new to them; something that has not been spoon-fed, but that they have honestly discovered for themselves—preferably something that you, yourself, have never heard of. I know that at first it is frightening to have your students know something you don't know, but it is the greatest achievement of a teacher to enable his students to surpass him.

I would like to close by citing the example of Galois, the great young French

mathematician. He died at the age of 21, yet he will remain for all time one of the great creative mathematicians because in his late teens he created entirely new ideas, the first really deep insight into group theory. This will remain fundamental mathematics for centuries to come. Galois's biography is more fascinating than any currently featured on television, but it is also very disturbing. He had to fight against traditional school curricula that straightjacketed thinking, and against teachers who neither understood nor had tolerance for the unorthodox mind. His good fortune was to be exposed to writings of Legendre, Lagrange, and Abel, and to find one high school teacher with the vision to encourage the young genius. What would be the fate of Galois today? Would he find anyone to encourage him in exploring entirely new paths, and would anyone help him find the mathematical literature that would inspire him, or would he be doomed to eternal boredom by being kept within the limitations of the traditional curriculum?

Have you read?

BICKNELL, G. G., "Mathematical Heredity," *Mathematics Magazine*, September-October, 1960, pp. 23–28.

By all means read this article. It is fascinating, but I am unable to do it justice without a complete rewrite. What is it all about? Magic squares. The Diabolic Four Square has progeny of other squares that obey the same properties. The order of the square can be raised to a family of super giants and still the same properties hold. Read it and marvel at the magic.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

CHURCHMAN, HENRY CLARENCE, "Inches and Meters Have No Quarrel," *The Duodecimal Bulletin*, August, 1960 (1174), pp. 15-1x and 25.

Will meters and inches disappear as did Noah's cubits? This is the question asked by the author of this interesting article. Effective July 1, 1959, by international agreement the inch became 2.54 cm. What did this do? It provided an easy convertor of inches and centimeters and this led to a more functional relation of the English and metric systems. Even after 120 years, the French are not wholly metric, many of the old units of measure stand.

Why not try a more usable system, the dozenal system or the dometric system? This system could be understood by both French and English. The basic unit could come from the earth's great circle and it could be called the dominaire. This would be subdivided by 12 and so on to our smallest unit. I guess that you had better read the article. I am lost in the dometrous.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

LANGFORD, C. D., "Children's Remarks," *The Mathematical Gazette*, February, 1960, pp. 2–3.

The sayings of children are often funny and sometimes sad, but they provide insights for the improvement of instruction. For example, the teacher said to Joe, "You should have learned this in the fourth grade." Joe replied, "I did learn it but didn't understand it so I forgot it," or the little girl who said, "Mathematics is not meant to be understood." What would you do with the little boy who looked blank when asked to multiply 3,000 by three, but when you wrote the problem on the board said, "Why didn't you say three times three and three naughts?" Perhaps you can add some of your own to the article and then see what it means to teaching.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

VON FOERSTER, HEINZ; MARA, PATRICIA M.; AMIOT, LAWRENCE W., "Doomsday: Friday, 13 November, A.D. 2026," *Science*, November 4, 1960, pp. 1291–1295.

You may say, what has such an article to do with mathematics? The only justification is quantity. We will not be bombed, we will not starve to death, we will be squeezed to death. On this date the human population will approach infinity. As is always the case, there are assumptions of fertility, environment, and the human factor. All of these relations can and are established by equations. Here it is, you decide what to do if anything from deleting the tax exemption for children to forbidding space travelers to re-enter our universe. Your students will be interested in this because it will be their problem.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

A report of a National Science Foundation summer institute in mathematics for high school students at Columbia*

GEORGE GROSSMAN, *William Howard Taft High School. New York, New York.*

Study of the programming of an automatic digital computer stimulates study of advanced topics in mathematics.

WHEN SUMMER NSF INSTITUTES for high school students were first announced by the NSF, the New York City Standing Committee on Mathematics, an arm of the Board of Education's High School Division, discussed possible topics and projects which would be of special value to selected high school students. Among the possible topics listed for the summer was "Programming for an Automatic Digital Computer."

The Science Honors Program had been, since 1958, sponsoring a series of Saturday classes for talented secondary-school students interested in the sciences and in mathematics. Through grants made by the Hebrew Technical Institute to the Columbia University School of Engineering, the Joint Program for Technical Education had been established "to advance the teaching of science and mathematics in secondary schools" and "to supplement for a select group of highly gifted students the regular science courses in high school, and add that element of creative and rigorous investigation which the development of scientific gifts requires but which conventional school programs rarely have."

* A talk prepared for the NCTM April, 1960 meeting at Buffalo.

A summer institute in "Advanced Mathematical Concepts and Computer Theory," a classroom and laboratory course in mathematics, logic, and electronic data processing, was planned for 32 students. The following topics were included in the proposed syllabus:

I. Programming for the IBM 650, using the Watson Laboratory of IBM, one block from the campus, with shorter units on the IBM 305 and 704, was planned.

Among the mathematical concepts to be developed around programming as the core are *scales of notation*, especially binary and biquinary. (The latter, similar to the abacus, is needed for the 650, binary and octal for the 704.)

Another concept is evaluation of complicated algebraic expressions—order of operations, with nests of parentheses finally coming into their own, for in programming polynomial evaluation these nests of parentheses are more efficient and turn out to be synthetic division in disguise!

Also to be developed are iterative processes—evaluation of roots, square roots, and higher roots by Newton's Method and the finding of irrational roots approximately, also by Newton's Method. Some

of the background in analytic geometry of lines and the differential calculus of polynomials is planned for. Solution of sets of linear equations leads to a study of *determinants*, *vectors* and *matrices*. The sum and product of two vectors and of two matrices lead to interesting programming problems, and matrix inversion leads to solutions of sets of equations available for programming.

The Theory of Numbers offers interesting problems for programming, and this gives us an opportunity to introduce these students to this wonderful branch of pure mathematics. Prime numbers, perfect numbers, amicable pairs, sum of divisors of numbers, divisibility, Pythagorean triples, Diophantine equations, and continued fractions, offer opportunity for exploration and discovery to students working at this level.

The evaluation of rapidly convergent infinite series will be undertaken to prepare tables of powers of e, and of sines and cosines. The trapezoidal and Simpson's Rules yield methods of evaluating π and natural logarithms. The areas under curves as definite integrals will be first developed informally. *Numerical analysis* is to be discussed throughout the course and interpolation and the method of differences are to be applied to the tracking of an artificial satellite.

II. Boolean algebra and its applications, including the algebra of switching circuits, developed inductively, symbolic logic and the algebra of sets, including the Commission on Mathematic's experimental text's approach to the definition of the probability of an event are to be introduced. Following abstract Boolean algebra, we plan to develop other abstract mathematical systems, including groups, rings, integral domains, fields, and linear vector spaces, interweaving these with a careful treatment of number systems to be described below.

The stated objectives therefore would be to introduce the superior mathematics student to significant branches of modern mathematics and to types of reasoning he normally would not encounter in high school or even college; and to present an important career possibility to his imagination.

Under an NSF grant of $33,800, three summer institutes, including the Mathematics Institute, were authorized. Each high school Principal in the areas selected was asked in May and June to choose his two most promising students, and a few large schools with selected student populations were permitted to nominate more than two students. The examination used was the Pre-Engineering Ability Test, Form ZPA, published by the Educational Testing Service, Princeton, N.J. This was not an engineering test, but a very searching multiple-choice test, dealing with the comprehension of scientific materials and with general mathematical ability. The actual selection was made on the basis of this test, with weight being given to the possible effect on certain schools, using the experience gained from the 1958 administration of the same test.

Thirty-five students were finally selected, including five girls; nine students were from the 99th percentile, nine from the 98th, eleven from the 97th, one from the 96th, three from the 95th, and two from the 90th percentile on the Pre-Engineering Ability Test.

For six weeks, from July 6 through August 14, these 35 students met, officially from 10 to 12 noon and from 2 to 4 P.M. daily, five days per week, in the mornings at Columbia and in the afternoons at IBM's air-conditioned Watson Laboratory one block away. But most of us were there at 8:30 A.M. daily at Watson Lab, punching out and testing programs. We adjourned to the School of Engineering for our 10 o'clock lecture, and many of the students stayed at the available libraries or at Watson Lab after 4 P.M.

We began with this selected group of 35 students, expecting the usual attrition. On August 14 we found that we still had our 35 students!

Average daily attendance was at least 34.5! We all found it an exciting experience, I in discovering how much these selected students could absorb, and my students in absorbing, exploring, and learning.

Our first objective was to learn IBM 650 programming. We chose the IBM 650 first because it was available to us; second because once programming for *one* stored program automatic digital computer has been learned, other computers are very easily mastered. Third, the IBM 650, using the denary scale with ten-digit "words" both for instructions and for data, is easy for students to comprehend and follow, especially when we have a desk calculator, also with ten digits for data and twenty digits for accumulators. This visual aid is especially helpful at the beginning in following step by step what the 650 is doing.

During the institute we learned programming not only in "machine language" but also in the Bell interpretive system, an excellent illustration of how one machine may simulate another type; in SOAP, a symbolic assembly program which makes the preparation of the coding for the program much less tedious, by relieving the coder of the problem of assigning storage locations as he proceeds; and in Fortransit, an automatic programming method where, for example, we tell the 650 to calculate a complicated algebraic expression and the 650 writes its own coded program of instructions. At the beginning we worked in machine language, learning how to take simple problems, analyze them, plan the numerical analysis, prepare the flow chart, code the instructions, punch the IBM cards, inspect them and the complete program to avoid wasting expensive machine time on the 650. Auxiliary IBM machines were available to us and learned by our students—key punches, interpreters, printing machines.

I mentioned above some of the types of problems we programmed. When we came to the evaluation of π, we used the area under the curve $y=1/(1+x^2)$ between $x=0$ and $x=1$, which equals $\pi/4$. Using Simpson's rule we divided this area first into 10, then 100, then 1000 subdivisions, expecting to get more accurate results with the greater number of subdivisions. We did get a closer approximation to π for $n=100$ than for $n=10$, but not from $n=1000$! For $n=1000$ we found our result was not even as good as for $n=10$, and the reason for this led us into a well-motivated discussion of truncation errors. We realized that 1000 smaller errors can add up to a greater cumulative error than 100 slightly larger errors. This introduced the importance of *numerical analysis*, a topic which these students will look for as they plan their college careers.

In the Theory Of Numbers, methods of discovering lists of prime numbers with their rich history were first introduced, then the sum of the divisors of numbers led to searching for perfect numbers and for pairs of amicable numbers. One of our most interesting problems was Goldbach's Conjecture. Of course we could not prove it by locating more and more pairs of primes whose respective sums are consecutive even numbers, and we couldn't hope to find an even number which did not have this property. But, by giving the student the problem of testing for primality, we changed the problem to find *the number of ways* in which each even number can be expressed as the sum of two primes. You may not know that 390 can be so expressed in 27 ways, that the largest number in our search which can be (or seems to be) expressible as the sum of two primes in only one way is 12, in only two ways the number seems to be 68, in only 3 ways it seems to be 128—all making Goldbach's conjecture more probable. We developed an interesting table, and I hope that some of our students are stimulated to spend more time with prime number theory and the theory of numbers in general.

Naturally not all 35 of our students became excited about programming. But for the 15 to 20 who did, programming may have opened their eyes to a possible

field of opportunity for the future, a field which can be exciting and which can, I have heard, absorb all of the Ph.D.'s in Mathematics for a long time to come. Their college courses may be planned with this as a possible objective.

Programming can be considered vocational, if handled that way, but we considered it mainly as a *core* around which to introduce these students to newer fields of mathematics and to newer methods of reasoning, because in planning efficient programs, interesting problems of coding lead to ways of reasoning new to high school students. And, most important, it served as a springboard for the rest of our course when I asked how a machine could do arithmetic electrically.

This led directly to the algebra of switching circuits, from there to other Boolean algebras, including the algebra of symbolic logic and the algebra of sets. We took time to explore the contents of our two texts—Allendoerfer and Oakley's *Principles of Mathematics*, and the Kemeny, *et al.*, *Finite Mathematical Structures* —on these topics. We discussed abstract Boolean algebra, using Huntington's postulates, noted parallelisms between the binary operations defined in switching circuits and in symbolic logic, introduced the concept of *isomorphism* between two sets under two binary operations and an equivalence relation, noted symbolic similarities between these algebras and the algebra of our number system, and then turned to our number systems.

We began with Peano's postulates for the natural numbers, proved the associative and commutative properties of the defined binary operation of *addition* and referred the students to other theorems to be proved, some as exercises for them, some for reading from Landau's *Foundations of Analysis* and/or other references. We next defined the set of *integers* as the set of ordered pairs of the previously defined natural numbers, whose properties we had established, defining $=$, $+$, $\times$ for these ordered pairs, first taking time to discuss thoroughly the concept of *equivalence relations*. It took at least a week before one of our students showed that reflexivity *was* independent of the other two requirements of symmetry and transitivity. (I still remember the morning when one of our group placed a system of ordered pairs, represented graphically, in the manner of the Illinois Unit 5, for which reflexivity did not hold for one number pair while the other two requirements were valid for the entire set.) We inspected all kinds of relations, categorized 27 different possible types in terms of *reflexivity*, *symmetry*, *transitivity* (always, never, or sometimes) (yes, no, or maybe for each), and impressed the importance of the *yes*, *yes*, *yes* relation, which we call the *equivalence relation*. Thereafter, whenever we defined a new system, and we defined many sets of ordered pairs—first integers, next rational numbers, later complex numbers—in each case we investigated the defined equivalence relation to prove, informally or formally, that it did satisfy all three requirements. We treated the defined binary operations which we denoted by $\oplus$ and $\otimes$ with a circle around each to discover how many properties we ordinarily expect from a "$+$" and a "$\times$"—commutativity, associativity, identities, inverses, one or two distributive laws, closure. Over and over again the concept of *isomorphism* came up. The subset of integers isomorphic to the natural numbers, each with its defined definitions of $=$, $+$, $\times$; the subset of rationals isomorphic to the set of integers; and later the subset of complex numbers isomorphic to the real numbers.

When studying properties of the set that we called *integers*, we defined an integral domain; when studying the rationals, we defined a number field; later we defined groups, rings, and linear vector spaces. And in all cases we proved properties of each system: why $a+b=b+a$; why minus times minus equals plus; why $a\times b=0$ implied either $a=0$ or $b=0$ or both, and when it didn't! Congruences were introduced to give us finite sets that were

groups, rings, and fields. Throughout we used the symbolism of the algebras of symbolic logic and of sets.

We next explored hypercomplex numbers, first quaternions, then matrices. I still look back on the morning when we proved the associativity of the multiplication of matrices, using double summations and observing, though not proving formally, the theorem permitting the interchange in the order of summations. The light in every student's eye was worth the effort!

For reasons that I think are apparent we did not have time to develop carefully the real number system, but we spent two hours on the definitions of real numbers given by Cantor and Dedekind and Ritt's treatment of infinite decimals. References were given so that interested students could explore further when they had time to do so.

One of the major objectives of the summer program turned out to be the introduction to these students of much of the literature of mathematics available to them both in libraries and in inexpensive editions, such as the Dover reprints. The two texts given to the students were satisfactory of course, but this Spring when giving a similar course to Science Honors Program students, I decided to experiment. Instead of purchasing 35 copies of a single book for all students, we bought a few copies of each of many books and used them as a circulating library, the students interchanging books every few weeks. One day during the summer I gave the students copies of the Dover catalog and spent some time describing each of the books in the catalog that I thought these students should explore. Many of our students purchased some of these books from the Columbia bookstores, and building a home library and the learning *how to read* mathematics books became important by-products of this institute. In the the present course I make these more than just a by-product.

The World of Mathematics and the *Encyclopaedia Britannica* were referred to constantly because of their wonderful articles. Thanks to the *Encyclopaedia Britannica* I received tear sheets of articles on mathematics and logic, and where pertinent I took time to read excerpts. To learn to love, to read, to purchase, and to own mathematics books became an important objective of the summer program, and I hope all my students come to love, to read, and to own mathematics books, too.

I have tried to describe my part in our summer institute. In addition, the Watson Laboratory provided personnel to assist our students in the machine room and one of their finest instructors, Mr. Eric Hankam, who gave some lectures to our students. Mr. Charles Freiman from the engineering department of Columbia alternated with me, his specialty being the Electrical Engineering aspects and applications of Boolean algebra. In addition, he covered much of the Commission on Mathematics experimental textbook on Probability and Statistical Inference.

I think you will agree with me that our students had good reasons to want to come at 8:30 A.M. daily and stay later and later each day. They loved the summer, as I learned later from individual conversations with students, their parents, and some of their teachers.

At the end of the summer, the students were given a take-home examination, and when returning their answers, some of their letters were indicative of their excitement during the course. Some of their statements are quoted here and on the following page.

> Now that I have finished the final, I realize how much I have learned without realizing it. Not only did I learn a lot this summer, but I realized that mathematics is far more diversified than I realized. I have become even more interested in mathematics than I was before. The course helped me to make firmer my decision to study mathematics in college.

> The scope of my mathematical understanding was greatly increased by the program. Looking back, I realize the enormous amount of knowledge that was presented to our class in the short span of six weeks.

I found it very rewarding, and it has disclosed many areas of mathematics which I otherwise would not have been aware of and investigated. It is a worthwhile thing, and I hope that it will be continued.

A letter I received in December from one of the girls is also of interest:

On Monday, December 28th, there will be a reunion of last summer's advanced mathematical topics course. . . . I'm sure everyone will be happy to see you again.

I'm afraid that the NSF wasted its money on me. Whenever I ask my math teacher a question, he yells at me for wasting class time. I've finally decided that he likes girls who giggle and don't know anything. So now I act like a dope in class and get high marks on my report card. It's very discouraging.

Two other December letters tell of positive gains from summer work:

In response to your request that we write to you: my Westinghouse project consisted of 650 programs for double precision floating point arithmetic, and an imitation Bell Interpretive system for using them. They needed too much initial information to make treating them like ordinary subroutines practical. The programs I submitted were hand optimized and doubtless full of errors, but the flow charts were ones I had used during the summer.

I should like to take this opportunity to thank you and Columbia for the chance to participate in such a program. I feel that it has given me a much better understanding of modern mathematics. It's one of the best things that ever happened to me.

One student wrote a sonnet!

Aftermath

Now that our summer course has finished its sixth week,
A sad farewell we bid to numbers binary,
No more constant dreams of Boolean variables
Or methods McCluskian or Quinary.

No more RAU. Farewell to e and to i.
We'll miss our n-factorial and the all-inclusive sigma.
For some, it's our last look at the 650
And we have never conquered "overflow," the final enigma.

But can we really get away from those machines,
Geniuses of magnet and memory, nuts and bolts?
For Mr. Freiman not only taught us probability
But how to have fun with 600 volts.

Don't just stand there contemplating—
Do it again! I'm fibrillating!

And one wrote new words to a well-known Gilbert and Sullivan ditty, beginning with: "I am the very model of a digital computer . . . " in which he continued,

A digital computer is a fascinating thing to be
For I can differentiate and integrate numerically,
I'll calculate the payroll of a giant corporation or forecast the results of all elections in the nation,
I'll translate English into Russian, Greek, or Transylvanian
Or Arabic or Sanskrit, Urdu, Chinese, or Rumanian,
I know the names of criminals, I also know their races,
And I can find both e and π to seven-hundred places.

When three of these students named me as "the one person most influential in the development of my career" in their Westinghouse Talent Search, each of these being among the list of 448 honors-award winners, a list which included 26 of our Science Honors Program students, I was even more proud, and I am sure no one will blame me!

What should a summer institute in mathematics for pre-seniors offer? These students *are* returning to high school after the summer, so a course in the Calculus and Analytic Geometry does not seem to me to be the correct course for them. For unless they use the calculus during their senior year, by the time they reach the advanced placement examination they may have forgotten too much, unless they repeat much of the summer's content—and they could have just taken the advanced placement calculus course during the senior year—so why should we use the summer months for it?

A summer institute like this should give the student something different from what he will get in the senior year or even in the freshman year at college. A course in the Foundations of Mathematics such as the one outlined gives the student an enriching experience which will *add* to his understanding of courses, whatever they be, taken from then on. These students may not be mature enough to go through Principia, but they certainly are mature

enough to discuss Gödel's Proof and its implications, to learn the requirements for postulational systems, to absorb abstract postulational systems, and to appreciate the nature of our number systems. More attention could have, and perhaps should have, been given to the theory of limits while developing the Cantor Theory of real numbers. Next time, I would try to spend more time with the theory of limits.

Another type of course which can be useful and which meets the criterion that it will not present ahead of time what *will* be covered in later courses these students are expected to take, is a course in modern applications of mathematics, with computer theory, statistical inference, and linear programming as possible topics. Our course tried to include not only computer programming but also, by considering switching circuits and Boolean algebra, how computers can be designed.

To open the eyes of these pupils to what mathematics is, and how modern mathematics is applied, seems to me to be an important addition that Summer Institutes can perform, to enrich and not to conflict with the school-year courses. In this we think we were successful. Two problems, however, remain to be solved: First there is that presented by the student—a charming young lady—who was encouraged by her summer course to be critical of mathematics and to ask questions that her teacher was incapable of answering or found untimely. Second, for me, the problem is how to return to normal classes after a summer with students who were not tied down by traditional, obsolete courses of study. I think that if some school system were to offer me only such stimulating classes it would be a dream come true.

What's new?

BOOKS

College

Analogue and Digital Computers, A. C. D. Haley, and W. E. Scott, editors. New York: Philosophical Library Inc., 1960. Cloth, viii+308 pp., $15.

Probability: An Introduction, Samuel Goldberg. New York: Prentice-Hall, Inc., 1960. Cloth, xiii+322 pp., $7.95.

The Teaching of Secondary Mathematics (3rd ed.), Charles H. Butler and F. Lynwood Wren. New York: McGraw-Hill Book Company, 1960. Cloth, ix+624 pp., $7.50.

Secondary

Applied General Mathematics (2nd ed.), Edwin B. Piper, Randolph S. Gardner, and Joseph Gruber. Cincinnati: Southwestern Publishing Co., 1960. Cloth, x+566 pp., $3.40.

Basic Mathematics for Electronics (2nd ed.), Nelson M. Cooke. New York: McGraw-Hill Book Co., 1960. Cloth, xii+679 pp., $7.50.

High School Mathematics: Unit 5—Relations and Functions, University of Illinois Committee on School Mathematics. Urbana, Illinois: University of Illinois Press, 1960. Paper, student's edition, vii+278 pp., $1.50; teacher's edition, $3.

Mathematics 9, experimental text, Ontario Mathematics Commission. Toronto, Canada: Service and Smiles Ltd., 1960. Paper, viii+394 pp., $2.

Miscellaneous

Accelerators, Machines of Nuclear Physics, Robert W. Wilson and Raphael Littauer. New York: Doubleday and Company, 1960. Paper, 196 pp., 95¢.

Mathematical Snapshots (new ed.), H. Steinhaus. New York: Oxford University Press, 1960. Cloth, 328 pp., $6.75.

Michelson and the Speed of Light, Bernard Jaffe. New York: Doubleday and Company, 1960. Paper, 197 pp., 95¢.

The Number Story, Herta Taussig Freitag and Arthur H. Freitag. Washington: National Council of Teachers of Mathematics, 1960. Paper, 76 pp., 85¢.

The Universe at Large, Hermann Bondi. New York: Doubleday and Company, 1960. Paper, 154 pp., 95¢.

The Water Shed, a Biography of Johannes Kepler, Arthur Koestler. New York: Doubleday and Company, 1960. Paper, 280 pp., 95¢.

The development of a concept: a demonstration lesson

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"Are you willing then," asked Socrates, "that we should make a delta on this side, and an alpha on that?"

THE METHOD OF INSTRUCTION which is currently emphasized for mathematics teaching is one which leads the student to develop a concept. The dialogue below is an adaptation of a lesson the writer has taught. It illustrates how students can be guided by questions and illustrations to arrive at a generalization.

TEACHER: In this age of space travel, important problems are the location of a point in space, the distances between points, and the directions of travel. Let us remember that in space there is no east or west, no up or down. We will need to locate points in space as we have done in graphing points on a plane in this class. Our lesson today will deal with the distances between points in space.

When two points are placed on a line, a simple measurement is generally used. This is illustrated by the way we use a ruler to measure the distance between these two points on the board. Now these two points may be placed anywhere on a plane, as is done in plotting airline distances on a map. To find the distance between two spaceships we need methods for determining distances between two points in space. First, let us deal with methods for determining the distance between two points on a plane. It is important to review some of the characteristics of the number line which we developed in the last unit. What are some of the characteristics of this number line?

STUDENTS: Any number whose graph is to the left of the graph of another number is less than that number. Any number whose graph is to the right of the graph of another number is the greater number.

A line can be used to graph solution sets of equations and inequalities.

Numbers from the sets of real, rational, integral, and natural numbers may be graphed on a line. For every point on the line there is a corresponding real number, and for every real number there is a corresponding point on the line. The distance between two points can be determined by subtracting the two co-ordinates. This distance may be directed or undirected.

TEACHER: Instead of a number line, suppose our grid is expanded to a plane? What must we now know to graph any point in the plane?

STUDENT: Two co-ordinates.

TEACHER: Can you give an example?

STUDENT: The point must be located by a horizontal and a vertical distance from some arbitrary starting point. The point (3, 5) is really here on this graph.

TEACHER (*pointing to* 5, 3): But isn't this point also (3, 5)?

STUDENT: Yes, but the horizontal distance is always the first number given.

TEACHER: Does this mean that we must make some mention of order of numbers in your earlier answer?

STUDENT: Yes, the two co-ordinates must be given in a certain order.

TEACHER: So now we are talking about an *ordered* pair of numbers. You have seen a grid similar to this in your ninth-grade work. Let's see if we can identify the parts of the grid you have used before.

[Students *identify the abscissa, ordinate,*

origin, X *and* Y *axes, quadrants, co-ordinates.*]

TEACHER: Let's recall how we graphed ordered pairs of numbers. Martha, would you come up to the board and point out some points on the graph? Class, let's give her some ordered pairs for points to be located. Be sure to pick points in each quadrant. (*This is done.*)

Does each point on this graph have an ordered pair of real numbers corresponding to it? Does this then include irrational numbers? What about fractional rationals? Can you give an example, George?

STUDENT: Yes, (3/2, 5/4).

TEACHER: As we look at this graph, let's see if we can discover some new properties or uses of it. Jim, would you come up to the board and plot a set of ordered pairs of numbers? Use (6, 0), (0, 8), and (0, 0). Label the first two P_1 and P_2 and the third point O. Now, can you find the length of line P_1O?

STUDENT: Yes, six units long.

TEACHER: How about line segment P_2O?

STUDENT: P_2O is 8 units long.

TEACHER: How did you arrive at that?

STUDENT: I counted the number of units on the axis.

TEACHER: What if P_3 was (0, 3867), which would make counting impractical?

STUDENT: We could use the same method we discovered about distance on the number line.

TEACHER: What was that?

STUDENT: Subtracting one co-ordinate from the other.

TEACHER: Very good! Now, how would you find the distance from P_1 to P_2?

STUDENT: I would use the Pythagorean Theorem because the distance from P_1 to P_2 is the hypotenuse of a right triangle.

TEACHER: What would you have?

STUDENT: I would have $6^2+8^2=x^2$, or $x^2=100$. The distance P_1P_2 is 10.

TEACHER: Under what conditions would this method always work in determining the distance between two points?

STUDENT: When these points are on the axes of the graph.

TEACHER: How do we know?

STUDENT: A right triangle will always be formed with the axes when a line is drawn between P_1 and P_2.

TEACHER: Suppose we move the points off the two axes. Is there any way by which we now can find the distance between points P_1 and P_2? Let's take P_1 as (2, 15) and P_2 as (7, 3). Any suggestions?

STUDENT: You could construct perpendiculars from P_1 and P_2 to the X axis and Y axis, respectively. These lines meet to form a right triangle.

TEACHER: In this case, what are the co-ordinates of the vertex of the third angle of the triangle?

STUDENT: The co-ordinates are (2, 3).

TEACHER: Could this ordered pair of numbers be determined without going through the construction process?

(*No response.*)

TEACHER: Compare the members of this ordered pair and the co-ordinates of P_1 and P_2.

STUDENT: It looks like the co-ordinates could be obtained by using the first co-ordinate of one point and the second co-ordinate of the other.

TEACHER: Good! Will knowing this third ordered pair help us determine the distance P_1P_2?

STUDENT: Yes, because now we can apply the Pythagorean Theorem to the newly formed triangle.

TEACHER: Will you do it, George?

[Student *writes:* $5^2+12^2=x^2$; $169=x^2$; $13=\bar{x}$.]

TEACHER: That's fine, but let's go back and see where you got your figures. How did you decide that 5 was the length of line P_2P_3?

STUDENT: They are 5 units apart.

TEACHER: How did you arrive at that?

STUDENT: I counted them.

TEACHER: Is there any other way of determining that distance?

STUDENT: Yes, by using the same method we used on the number line.

TEACHER: You mean by subtracting the two co-ordinates on the line?

STUDENT: Yes.

TEACHER: So now we could show how the 5 was obtained by writing, instead of 5^2, $(7-2)^2$. What about the length of P_1P_3?

STUDENT: That could be written in the same fashion, $(15-3)^2$.

TEACHER: Now, George, if we use these values in your equation in place of the 5^2 and 12^2, what are the results?

STUDENT: $(7-2)^2+(15-3)^2=x^2$.

TEACHER: What is P_1P_2 now?

STUDENT: $\sqrt{(7-2)^2+(15-3)^2}=d(P_1P_2)$.

TEACHER: Can you see any relationship between the co-ordinates of P_1 and P_2 and the equation we have just developed?

STUDENT: They are the same.

TEACHER: What do you mean, "the same"?

STUDENT: The 7 and 2 are the first numbers of the ordered pairs, and the 15 and 3 are the second numbers of the ordered pairs.

TEACHER: Suppose we want to generalize from this case to a situation where P_1 and P_2 are any two points. What would the co-ordinates of these points be?

STUDENT: P_1 could have co-ordinates (x_1, y_1) and P_2, co-ordinates (x_2, y_2).

TEACHER: How does our equation look?

STUDENT: It would look like this—

$$\sqrt{(x_2-x_1)^2+(y_1-y_2)^2}=d(P_1P_2).$$

TEACHER: Are we concerned with directed or undirected distance?

STUDENT: Undirected distance.

TEACHER: That's right. And you recall the undirected distance, $(P_1P_2)=|x_2-x_1|$. So in our equation,

$$\sqrt{|x_2-x_1|^2+|y_1-y_2|^2}=d(P_1P_2).$$

From what we have done thus far, can you see any way of obtaining distance P_1P_2 without constructing the graph or completing the triangle?

STUDENT: Merely replace the co-ordinates given in the problem in the formula and solve.

TEACHER: Let's try it on the set of ordered pairs, $P_1(15, 4)$ and $P_2(7, 19)$. You solve it at your seats, while I set it up on the board. (*Does so.*)

Do you all get the length 17 for P_1P_2? How can we check this length?

STUDENT: Construct the triangle on graph paper.

TEACHER: Is there any reason why this length needs to turn out to be an integer? a rational number?

STUDENTS: No.

TEACHER: That's right, we have already constructed line segments of definite length which are irrational. We might find the distance to be $\sqrt{2}$ or $\sqrt{23}$, etc.

It appears that we can find the distance between any two points on a plane. Would this process work if the points P_1 and P_2 were in different quadrants? What happens when one of the points lies on the origin of the graph? What facts about geometric figures can be developed by using this principle?

These questions are among those you should be able to answer after working out solutions to the following problems.

PROBLEMS

1. On the co-ordinate axes, locate $P_1(1, 1)$ and $P_2(3, 4)$. Draw the line joining the two points. Complete the right triangle and show the lengths of the two legs. Use the Pythagorean Theorem to find $d(P_1P_2)$. Show that this length can also be obtained by using the distance formula.
2. Find the distance between $P_1(-30, +33)$ and $P_2(+6, -8)$, using the distance formula.
3. Find the distance between $P_1(8, \sqrt{3})$ and $P_2(-6, \sqrt{3})$, using the distance formula.
4. Find the distance between $P_1(0, 0)$ and $P_2(5, 3)$, using the distance formula.
5. Show that the points (8, 1), (−6, −7), and (2, 7) are the vertices of an isosceles triangle.
6. Find the distance between $P_1(0, \sqrt{12})$ and $P_2(0, \sqrt{3})$.

Extra Credit. Develop a distance formula which can be used to find the distance between two points in space.

• **EXPERIMENTAL PROGRAMS**

Edited by Eugene D. Nichols, Florida State University, Tallahassee, Florida

Improved programs in mathematics require inservice education for teachers

by Kenneth E. Brown, U.S. Office of Education, Washington, D.C.

Introduction

This article is a report of the conference held March 17–19, 1960, at the U.S. Office of Education on "Inservice Education of Teachers of Secondary-School Mathematics," under the joint sponsorship of the Office of Education and the National Council of Teachers of Mathematics.

A comprehensive bulletin on the proceedings of this conference is available from the U.S. Office of Education.

The conferees agreed that the need for developing realistic programs for teacher preparation in mathematics is *crucial.* The inservice education should involve cooperative efforts of teachers, supervisors, administrators, colleges, state departments of education, and program-sponsoring foundations.

The brief report summarizes for the administrators of secondary schools what is being done, and what can be done to improve the inservice education of teachers of mathematics.

Why changes are needed in the mathematics curriculum

Space missiles are only symbols of the great explosion of the twentieth century of scientific knowledge. One of the most important factors contributing to this explosion is the revolutionary advance in both the development and the use of mathematics.

Not only are new requirements being placed on mathematics in the fields of physics, chemistry, and engineering, but in other fields mathematics is being put to new and even more astonishing uses.

The biologist is applying mathematical theory to the study of inheritance; industry is using mathematics in scheduling production and distribution; the social scientist is using ideas from modern statistics; the psychologist is using mathematics of game theory. In fact, the logic of mathematical models shows promise as the basis for developing teaching machines for all areas of knowledge. The new uses of mathematics require less manipulation of formulas and equations but greater understanding of the structure of mathematics and mathematical systems. There is less emphasis on human computation that can be done by machines, and more emphasis on the contruction of mathematical models and symbolic representation of ideas and relationships. Because of these new uses, mathematics is being firmly woven into the fabric of the national culture. The role of mathematics is not only to grind out answers to engineering problems, but to produce mathematical models (prototypes) that forecast the outcome of social trends and even the behaviorial changes of the group. Such important new uses and interpretations of mathematics require that students have a program with a

greater depth than the classical program designed for nineteenth-century education. The demands of society require a thorough revision of our present secondary-school mathematics curriculum.

Reasons for changes in the mathematics curriculum

1. The subject matter of mathematics is constantly growing, not only in the area of advanced mathematics, but also in the area of elementary mathematics.
2. Mathematics today is being called on to meet a wide variety of needs of which we had not dreamed a few years ago.
3. The emphasis in mathematics is changing significantly: it is moving away from human computation to an understanding and construction of symbolic representations of factors that relate to scientific or social situations.
4. New mathematical ideas, language, and symbolism are being introduced to give better understanding of the subject.
5. There is a growing realization of the need for better articulation between secondary mathematics and college mathematics.
6. There is a new awareness that some of the mathematics now being taught in high schools is obsolete and should be replaced by more significant subject matter.
7. A number of educational experiments have demonstrated the feasibility and advantages of teaching new topics in high school mathematics courses.
8. Several national groups of educators have made detailed studies of possible curriculum changes with specific recommendations.

WHAT PROJECTS ARE UNDER WAY IN THE MATHEMATICS CURRICULUM?

The School Mathematics Study Group (SMSG)

For the first time in the history of education in the United States, more than a hundred college mathematicians and secondary-school teachers together have planned and written sample textbooks for grades 7 to 12. These textbooks have been tried out in many types of schools under competent teachers especially trained for the new task. This project of the School Mathematics Study Group was generously financed by the National Science Foundation.

The textbooks may be purchased from the Director of the Project, Dr. E. G. Begle, Drawer 2502A, Yale Station, New Haven, Connecticut.

The University of Illinois Committee on School Mathematics (UICSM)

Since 1951, the University of Illinois Committee on School Mathematics has carried on a strong program of study and experimentation in the improvement of school mathematics. Many schools have tried out the published materials.

Textbooks and teachers' manuals for UICSM first-year algebra and plane geometry may be purchased from the University of Illinois Press, Urbana, Illinois.

The Commission on Mathematics (CEEB)

The Commission on Mathematics of the College Entrance Examination Board, established in 1955 and composed of mathematicians and high school teachers, has made recommendations for major improvements in the secondary-mathematics curriculum. The Commission has issued publications which may be had by writing to the College Entrance Examination Board, Educational Testing Service, Box 592, Princeton, New Jersey.

The University of Maryland Mathematics Project (UMMaP)

The Maryland Project has thus far concentrated on mathematics in the junior high school.

Published materials may be purchased from Dr. John R. Mayor, University of Maryland, College Park, Maryland.

Ball State Teachers College Mathematics Program

This experimental program provides new material in algebra and geometry. Published materials can be obtained from Ball State Teachers College, Muncie, Indiana.

Boston College Series

Under supervision of the Rev. Stanley Bezuszka, S. J., materials are being planned for grades 9–12 in the Boston College Mathematics Series. The emphasis is on the structure of mathematics approached from the historical point of view. Inquiries about the program should be sent to the Rev. Bezuszka.

Secondary-School Curriculum Committee

The Secondary-School Curriculum Committee, under the direction of Mr. Frank B. Allen, was appointed by the National Council of Teachers of Mathematics to study the mathematics curriculum and instruction in secondary schools with relation to the needs of contemporary society. The committee report, entitled "The Secondary-Mathematics Curriculum," and special reports, such as "The Supervisor of Mathematics, His Role in the Development of Mathematics Instruction," may be purchased from the National Council of Teachers of Mathematics.

Other projects

In addition to the foregoing projects, many local and state experiments and studies are under way. For example, a survey of a randomly selected sample of one-fifth of the public secondary schools indicated that 40 per cent of these school administrators planned to revise the mathematics curriculum.

Curriculum changes recommended

Examples of recommendations of the groups listed are:

1. Include the elementary concepts and language of sets.
2. Teach a more refined concept of function.
3. Strengthen the logical development of geometry.
4. Present in plane geometry some of the elements of analytic geometry and solid geometry.
5. Modernize the vocabulary of elementary algebra.
6. Study inequations as well as equations.
7. Stress understanding rather than manipulation.
8. Cultivate an understanding and appreciation of the structure of mathematics.

Inservice education for teachers

Do teachers need inservice education for the new mathematics curriculums?

Many educators are realizing that inservice education is a continuing need for all teachers if they are to remain abreast of new developments in subject matter and teaching methods.

For example, the School Mathematics Study Group realized that high school teachers in the SMSG Experimental Centers needed specific training for teaching topics newly incorporated into high school curriculums. A college teacher was assigned to work with the teachers once a week.

The University of Illinois Committee on School Mathematics attaches so much importance to inservice education that it will not permit any teacher to participate in its project until he has spent a summer on the campus in preparation under its supervision. Afterwards, supervisors are assigned to visit participating teachers during the school year and assist them in the new type of teaching.

What inservice education programs are in progress?

A questionnaire sent by one of the conferees to more than 100 of the large colleges indicated that 62 per cent were giving inservice education courses on the campus and 40 per cent in various adjacent school

systems. Only 13 per cent of the college institutes were paid for by the college and were free to the teacher. These institutes were of short duration, like the one-day institutes at Northwestern University, University of Illinois, and Western Washington College of Education. Institutes for the re-education of mathematics teachers have been sponsored by foundations and industries, including the National Science Foundation, the Ford Foundation, Shell Oil Company, General Electric Company, Camille and Henry Dreyfus Fund.

More than half of the colleges indicated that they did provide consultative help to teachers upon request.

More than half of the states have held conferences on the inservice education of mathematics teachers. About the same percentage have published brochures or booklets.

In about half of the 121 large schools studied there were groups of mathematics teachers organized to study the mathematics program. Television programs at Buffalo, New York City, Denver, and Chicago have been used for teacher inservice education.

About one-third of the schools had organized an inservice education course which consisted of a college-level class to introduce the newer concepts in mathematics to the teachers.

How are administrators helping in the inservice education of the mathematics teachers?

The conferees reported many ways in which the administrators of secondary schools are helping teachers to upgrade their professional competence. Some of these are reported below. Additional suggestions are contained in the comprehensive report of the conference proceedings which is available from the U.S. Office of Education.

1. Encourage teachers to read and study books on modern mathematics. Examples: One school provided a list of such books which could be obtained through a simple telephone request. Niagara Falls public schools, New York, provides a professional library for teachers and encourages them to read and study the new books.
2. Develop a co-operative inservice education program between the public schools and a college or university in the area. Examples: The Dallas public school system has such a co-operative arrangement with the Southern Methodist University. The public schools of Syracuse, New York, and Syracuse University are also co-operating in this manner.
3. Provide the necessary funds to pay teachers to work on curriculum materials during the summer months. Example: The Eugene, Oregon, public school system is operating such a program.
4. Encourage teachers to visit classes in schools where the new materials in mathematics are being taught. Substitute teachers should be provided by the administration. Example: The Tucson, Arizona, public school system is providing this kind of encouragement and assistance to its teachers.
5. Set up salary schedules which provide appropriate increases for teachers who complete additional college courses in the subject matter of their teaching field.
6. Make provisions in the regular school budget for financing a continuous inservice education program.
7. Arrange the school schedule to allow teachers, *on released time*, to participate in seminars, curriculum study groups, professional organization meetings, and teacher inservice programs. One administrator of a large school system emphasized this point by stating, "It is our feeling that to reach all the mathematics teachers, we will have to provide inservice education on school time."
8. Encourage local industrial organizations and philanthropic agencies to sponsor inservice educational pro-

grams. Examples: The Timken Roller Bearing Company at Canton, Ohio, paid 50 per cent of the expenses incurred by local mathematics teachers in attending summer sessions at college. The local industries of Dayton, Ohio, paid for an eight-week institute for local teachers.

9. Encourage mathematics teachers to join their professional associations both in mathematics and in general education.
10. Assist teachers to develop professionally by encouraging them to take an active part in experimental mathematics programs.
11. Organize a mathematics-curriculum study committee consisting of teachers from various grade levels. This would help to provide continuous development of the basic unifying concepts of mathematics.
12. Engage well-qualified consultants to assist the local mathematics curriculum committee in their study of the present total school mathematics program.
13. Provide a teachers' reading and conference room with books on modern mathematics and the evolving new course materials.
14. Encourage the development of appropriate correspondence courses in mathematics for teachers. Each state department of education may assist one or more of the colleges in the state in the co-operative development of such courses.

For more complete information on inservice education of mathematics teachers, see the report on this subject from the U.S. Office of Education, Washington, D.C.

Miami area inservice programs for mathematics teachers

*by Jeff West and Agnes Y. Rickey, Dade County School System, Florida**

What is a good mathematics program in the space-age 60's? With all the varied programs being developed throughout the country today and the many alternative courses of action available, it is little wonder that teachers and school administrators find it difficult to develop an intelligent answer to this question. Here is an account of how one school system attempted to cope with this pressing problem.

The wheels first began to turn when consultants from the United States Office of Education visited the Dade County school system in the spring of 1960 and inquired about plans for a program of modern mathematics. Additional impetus was given by Dade County mathematics teachers who had studied under National Science Foundation grants during the summers at colleges and universities throughout the nation. Further help was given by the University of Miami through

* Dr. West is director of curriculum and instructional services and Miss Rickey is supervisor of mathematics for the Dade County, Florida, school system.

several inservice courses for mathematics teachers which were also sponsored by the National Science Foundation. Assistance in initiating a new program was still needed.

It was then that Dr. E. G. Begle, of the School Mathematics Study Group, spoke as a guest lecturer under sponsorship of the National Science Foundation and the University of Miami. At about the same time, Mr. Robert Kalin, of the Florida State University, was giving demonstration lessons using School Mathematics Study Group materials in several schools in the system. The impetus to try some of the "new" mathematics was gaining momentum.

A tentative plan for an experimental program was developed by principals and teachers working with the supervisor of mathematics. The director of curriculum and instructional services submitted this proposed plan to the superintendent's instructional staff and obtained permission to develop a pilot experiment in several seventh-, eighth-, and ninth-grade classes using the School Mathematics Study Group. At the same time, a budget request for the necessary instructional materials was submitted and approved.

Plans were developed for the department of mathematics education of the Florida State University to offer an extension class every Monday evening with Dr. Eugene Nichols as instructor. Dr. Nichols stressed fundamental mathematical concepts which provided teachers with a solid basis for presenting the new School Mathematics Study Group materials.

Wherever possible, two teachers on the same grade level in a school were encouraged to enroll in the class on Monday nights. This enabled one teacher to teach from the SMSG text while the other teacher who taught from the standard text was free to use as much of the SMSG material as she wished. It also made it possible for teachers to discuss their lessons with one another and, in the event of an emergency, the second teacher could take over for the SMSG teacher without loss of continuity in the course of study.

All of these plans seemed to insure a reasonable degree of success in the introduction of this new program, but there still remained one area which needed to be strengthened. It is one thing to lecture to teachers in a formal classroom situation about a new mathematics program, but it is an entirely different matter when this new program is forced to meet the real test of being taught to a group of energetic and inquisitive junior high school pupils. This problem was met head-on by Dr. Nichols, who planned a series of weekly visits to the classrooms during which he served as the teacher and personally demonstrated the most effective procedures for teaching the new materials. His approach placed a heavy emphasis upon allowing pupils to discover important mathematical concepts for themselves. These demonstration lessons were one of the most effective procedures used to develop clear understandings about the new content as well as its method of presentation.

It is recognized that the results of the Dade County Experiment will not supply all the answers concerning the opportunities and possibilities offered by the School Mathematics Study Group program. There are many other phases of this new approach to modern mathematics which are still in need of exploration. The results of this experiment should provide evidence that will serve as a basis for future planning.

Because of the tremendous number of mathematical discoveries which are daily being added to our total knowledge, and because there is a real need for an educated citizenry with a thorough understanding of our scientific age, it is of utmost importance that a continuous effort be made to modernize the mathematics curriculum so that teachers can more adequately prepare our boys and girls to meet the challenge of tomorrow.

Edited by Howard Eves, University of Maine, Orono, Maine

*Journey to Delos**

by Laura Guggenbuhl, Hunter College, New York, New York

In recent years, a number of delightful magazine articles [1]† have appeared telling about the beauties, the ancient history, and the friendliness of the Greek Islands. At what better time could one say that one of the most famous of the Greek Islands played host (even if only in name) to one of the most famous of the mathematical problems of antiquity?

As most readers no doubt well know, the Delian problem [2], namely the problem of the duplication of the cube, is one of the so-called three famous problems of antiquity, the other two being the trisection of an angle and the squaring of the circle. These three problems have never suffered for lack of a sponsor; in fact it was as early as 1775 that the Academy of Sciences at Paris [3] announced that it would consider no further solutions for the trisection of an angle. The problems, with solutions, have been discussed from many points of view and in the greatest possible detail by scores of mathematicians, from Hippocrates in the fifth century B.C. to Eves and Breidenbach in 1953, and Van der Waerden in 1954 [2]. And yet one cannot resist the temptation of adding a few words to the historical and geographical background of the problem of the duplication of the cube.

Let us embark then upon a brief journey to Delos [4], the legendary birthplace of Apollo and, at one time, the most important of all the islands in the Aegean. Today Delos is a port of call on many well-advertised tours, but early world travellers have left accounts of visits to the island which date back to at least the beginning of the fifteenth century. The history of Delos is divided into the four periods—Early Ionian, Athenian, Macedonian, and Roman. Note that as a world power Delos came to the end of its days around the time of the beginning of the Christian Era. The work of excavation and the archaeological research [5] on the island have been conducted, since the latter half of the nineteenth century, by the French School of Classical Studies at Athens. Many of the results of these investigations have been incorporated into the "Fouilles de Delos," and several of the references listed at the end of this article came from this source.

It is a little difficult to decide where we should begin the story of our journey. Perhaps our account should be scientifically encyclopedic and begin by saying that the geographical components of Delos are approximately 25 degrees east longitude and 37 degrees north latitude, and that the dimensions of the island are about three miles in length and one mile in width. Perhaps we should say first that Delos has been called the center of the Cyclades, a

* The author wishes to express her deep appreciation for the kindness which was shown to her at the Firestone Library of Princeton University, at the Art Reference Library of the Metropolitan Museum of Art in New York City, at the Library of the Academy of Sciences in Paris, and at the Niedersächsische Staatsund Universitätsbibliothek in Göttingen. Much of the research for this paper was done in these libraries.

† Numbers in brackets refer to the references given at the end of the article.

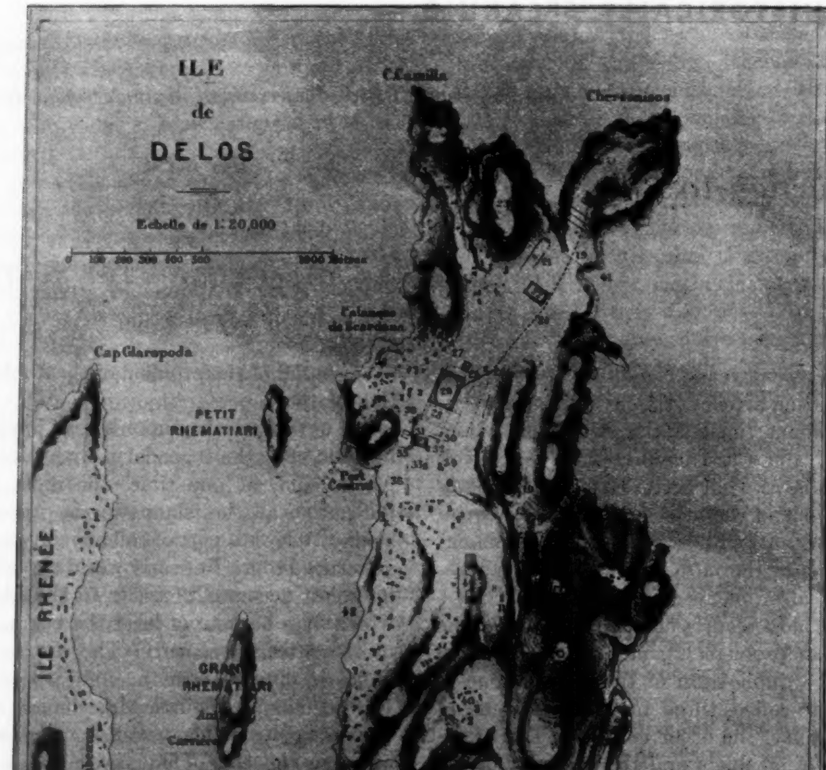

Figure 1

A nineteenth-century map of Delos from Lebègue, "Recherches sur Délos"

group of islands off the southeastern coast of Greece. Or perhaps we should tell about arriving on the Island of Mykonos, the "Capri of Greece," with its whitewashed houses and streets and its quaint old windmills, at 10 P.M., and being warned that if the winds were not favorable on the following day we might not be able to get to Delos. We had come a distance of about 4,000 miles to get a first-hand glimpse of the famous island, and if the winds were not favorable on the morrow we might not be able to make the hour trip from Mykonos to Delos. Fortunately for us, the following day was a most beautiful summer day with a gentle breeze and perfect skies. We boarded a large motor launch and, after a delightful ride, we stepped ashore on the sacred island of the ancient Greeks. No doubt it was on this very same spot that thousands of Greeks, Romans, and Saracens, scores of emperors, slaves, and free men, and pirates had first set foot on Delos. The soft sea spray had had a somewhat hypnotic effect, and it was not at all difficult to feel a sense of kinship with those bygone persons who had been on the island before us.

We had read that Delos was now uninhabited, in fact that no one had lived there for the past several centuries, and we knew that it would seem like a huge barren and deserted rock. We walked down the famous Avenue of the Lions, took pictures, and studied the fragment of the statue of the colossal Apollo as a flock of wild turkeys pecked their way disdainfully through the dried grass and marble ruins. Further inland, we saw the Temple of Isis, the House of the Masks, and other buildings with beautiful mosaics and tile floors which have been excavated and restored by members of the French School of Classical Studies at Athens. Strangely enough, the days when thousands of pilgrims came to worship on the island, when thousands of slaves were sold in a single day in its slave market, did not seem as remote as the record would have us believe.

Legend says that Delos, originally a floating island, was chained to the bottom of the sea by Zeus, to serve as a haven for Leto, who had been driven from place to place by the jealous Aphrodite. Apollo, the son of Zeus and Leto, was born at the foot of a palm tree on this tiny island. Present-day guides point, with pleasure, to the exact spot where this palm tree stood. The story goes on to say that Apollo left Delos for the mainland of Greece, and that he settled at Delphi on the Gulf of Corinth. However, Delos was a most prosperous community; its best period was about the second century B.C. It was a commercial and banking center, the seat of a most active slave market, and the center of the religion based upon the worship of Apollo. Religious festivals, Greek games, contests of strength and song, and chariot races were held here. At one time Delos was legally transformed into a truly sacred spot, and no one was allowed to be born or to die there. A woman about to give birth to a child, or a person about to die, was taken to another island close by. Many other measures were used to purify and sanctify the island.

The highest point on the island, about 350 feet above sea level, is the top of a hillock known as Mt. Cynthus. In the earliest times, a primitive temple to Apollo stood about halfway up on the slope of this hill. It is generally agreed that the famous oracle of Delos was housed in this temple. There is historical evidence to indicate that this oracle was consulted as late as the first century A.D. About this oracle and the Delian problem, Eves has this to say:

> Again, still later, it is told that the Delians were instructed by their oracle that, to get rid of a certain pestilence, they must double the size of Apollo's cubical altar. The problem was reputedly taken to Plato who submitted it to the geometers. It is this latter story which has led the duplication problem frequently to be referred to as the Delian problem. Whether the story is true or not, the problem was studied in Plato's Academy, and there are higher geometry solutions attributed to Eudoxus, Menaechmus, and even (though probably erroneously) to Plato himself.

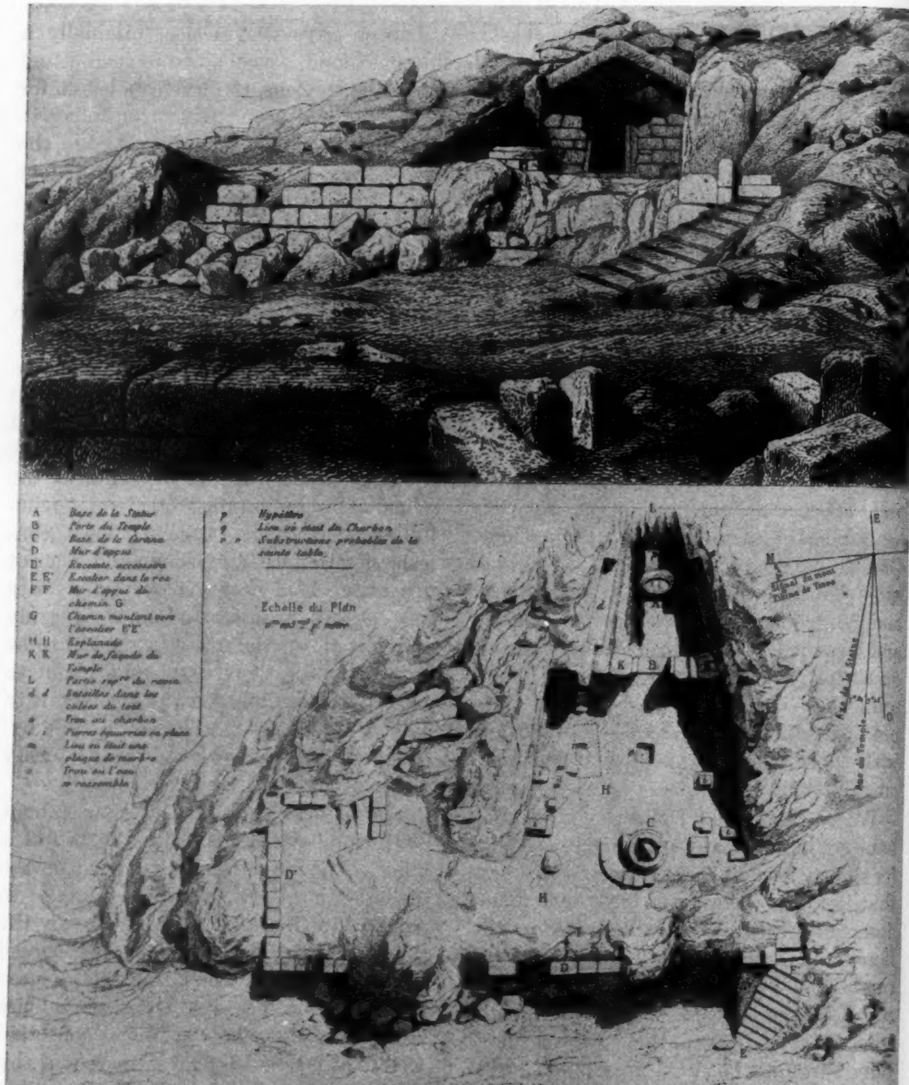

Figure 2

The famous oracle of Apollo at Delos from Lebègue, "Recherches sur Délos"

The upper portion of the sketch shows the sacred cave in a boulder-strewn section about half way up the slope of Mt. Cynthus.

The lower portion shows the plan of the sanctuary, the pedestal on which the statue of Apollo stood, and the road, stairway, and terrace dug out of the side of the mountain.

Thus the location of the oracle of this famous story is determined beyond the shadow of a doubt.

According to Lebègue [5], the cubical altar mentioned in the above story was situated in the Temple of Apollo in the sacred precincts of Delos on the shore of the island. However, on our visit to the island we saw no reference to, or ruin of, a cubical altar.

The decline of Delos as a commercial and religious center of the first order was not hastened by any catastrophe of nature such as the eruption of Mt. Vesuvius. Instead, the disintegration of this highly cultured society was a gradual process. Some of the destruction was due to ruinous warfare, and some to the attacks of hostile hordes from other parts of the Mediterranean. Ancient epigrams indicate that Delos was deserted at the end of the second century A.D. Unlike the situation which prevailed at many other pagan shrines, no Christian church was ever built on the ruins of any of the temples on Delos.

Just before it was time for us to leave, we paid a visit to the local museum set up by the French School of Classical Studies at Athens, and then we boarded the launch for the return trip to Mykonos. In a short time Mt. Cynthus was once again nothing more than a dim image on the horizon, bathed in the history of the

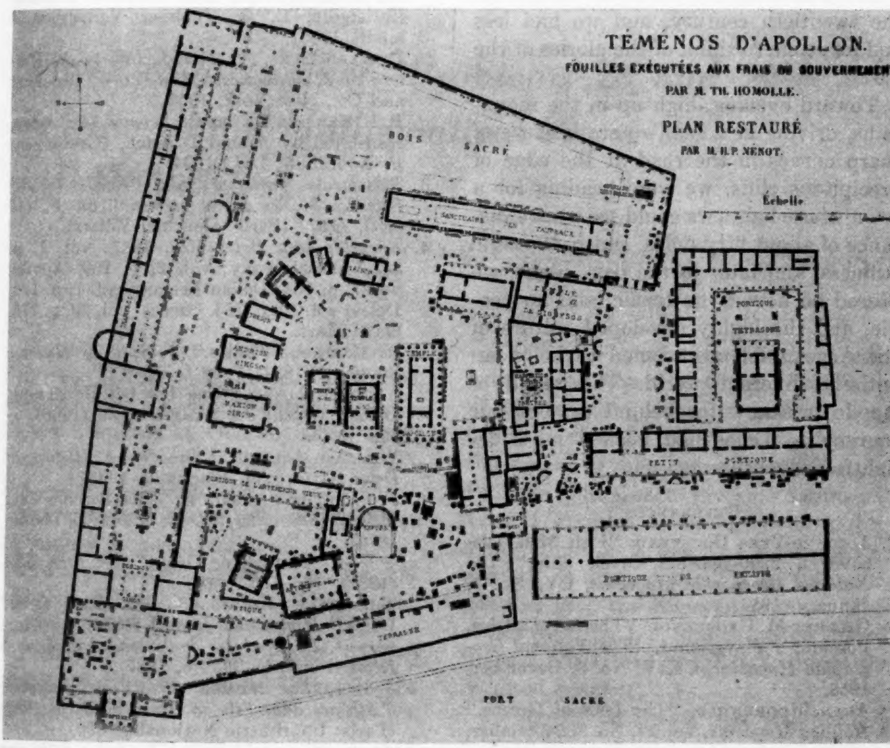

Figure 3

Plan of the sacred precincts of Delos on the shore of the island from T. Homolle,
Les travaux de l'école française d'Athènes dans l'ile de Délos

Note the Temple of Apollo near the center of the plan. According to Lebègue, it was here that the famous cubical altar of Apollo (which the Delians were advised to double) was situated.

glorious splendors of the past. On the following day we returned to the mainland of Greece.

From Athens we drove to Delphi [6], where the work of excavation and restoration is also being conducted by the French School of Classical Studies at Athens. The oracle of Apollo at Delphi far surpassed the oracle at Delos in importance, in spite of the fact that Delos was the acknowledged birthplace of Apollo. During the next few days we drove through the rugged mountain fastnesses of northern Greece, the legendary realm of Apollo during the six months of the year when he did not preside at Delphi. As we crossed the border into Yugoslavia [7], we were confronted by problems unique to the twentieth century, and we had less and less time to think of the glories of the past.

Toward evening, high up in the mountains, driving at a snail's pace because of sharp curves in the road at the edge of precipitous cliffs, we were heading for a town whose lights we could see from a distance of about fifty miles. But in the eerie stillness, and later in the dark night, we passed no houses, no traffic, and no people, and the highly developed culture of the Aegean Islands seemed to be in an entirely different world. The realm of Apollo was now far behind us, but our journey to Delos had been a most delightful adventure.

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"Think-of-a-Number" Problems 28 and 29 of the Rhind Mathematical Papyrus (B.M. 10057-8)

by R. J. Gillings, Sydney University, Australia

Section V of the *Rhind Mathematical Papyrus* (Chace, Manning, Archibald, published by the Mathematical Association of America at Oberlin, Ohio, 1927) consists of six "*AHA*" or *Quantity Problems*, which are numbered 24 through 29. The first four of these problems, numbers 24 through 27, are as follows:

Problem 24. A quantity and its 1/7 added together become 19. What is the quantity?

Problem 25. A quantity and its 1/2 added together become 16. What is the quantity?

Problem 26. A quantity and its 1/4 added together become 15. What is the quantity?

Problem 27. A quantity and its 1/5 added together become 21. What is the quantity?

The Scribe's "solutions" of these problems are set down by Chace (Vol. I, pp. 67–69), and the methods used are analogous to that commonly referred to as the *method of false assumption.* Thus, for Problem 24, Chace gives: "Assume 7. Then 1/7 added to this gives 8, and as many times as 8 must be multiplied to give 19, so many times 7 must be multiplied to give the required number. This comes to 2 1/4 1/8, and therefore since 7 times 2 1/4 1/8 is 16 1/2 1/8, this is the required quantity."

It should be noted that the Scribe's work in the original hieratic is much more abbreviated than is shown in Chace's translation. For example, the question "*What is the quantity?*", the phrase "*Assume* 7," and the statement, "*As many times as* 8 *must be multiplied to give the re-*

quired number," are all added by Chace to make the numerical calculations intelligible to the reader, and this applies to all four problems, the understanding of which is clear and does not raise any controversy as to their meaning or significance.

But Problems 28 and 29 present interesting issues, particularly as the Scribe has written them in an even more abbreviated form.

Chace's translation of Problem 28 is (Vol. I, p. 69):

> A quantity and its 2/3 are added together and from the sum 1/3 of the sum is subtracted, and 10 remains. What is the quantity?
>
> Subtract from 10 its 1/10 which is 1. The remainder is 9. This is the quantity. Its 2/3, 6, added to 9 makes 15, and 1/3 of 15 taken away from 15 leaves 10. Do it thus.

Chace attempts an explanation of this problem by quite reasonably assuming that the Scribe would have adopted the same procedure as he did in the four previous problems, namely that of *false assumption.* So he says (Vol. I, p. 70),

> It may be supposed that our author first solved the problem as follows:
>
> Assume 9.
>
> Then 9 plus 2/3 of 9 is 15, and 1/3 of 15 subtracted from 15 is 10.

And he proceeds to comment:

> As many times as 10 must be multiplied to give 10, that is, once, so many times 9 must be multiplied to give the required number, and therefore the required number is 9. But now he notices that 9 is obtained by taking away its 1/10 from 10, so he puts in the solution given in the papyrus. The solution does not seem to be complete.

It is with this explanation that I take issue. Let us now look at the original hieratic and the exact literal translation as given by Chace himself in Volume II, Plate 51, shown in Figure 1.

Chace's translation:

> 2/3 is to be added, 1/3 is to be subtracted, 10 remains. Make 1/10 of this, there becomes 1, the remainder is 9. 2/3 of it namely 6, is to be added, the total is 15, 1/3 of it is 5. Lo, 5 is what went out, the remainder is 10. The doing as it occurs.

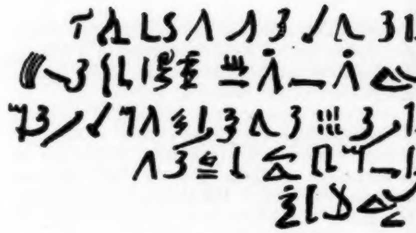


Figure 1: Problem 28

We find that Chace has added the first two words, *"A quantity,"* in the same sense as he did for the previous problems, that is, as if it were the modern *"x,"* the unknown number, or quantity, whose value is to be found. But I would interpret the Scribe's problem in this case as referring, not to some specific quantity as in Problems 24, 25, 26, and 27, but to *any number at all* that his students might think of, so that he is here showing them some of his "Obscure secrets" [1],* some of his "magic with numbers," or, as the modern teacher would say to his class, he says:

"Think of a number. Add 2/3 of this number to itself. Subtract 1/3 from the result you get. Now tell me your answer."

Suppose the first student announced his answer as 40. The Scribe then subtracts from this number its tenth part, getting 36, and this number, he says to the student, is "the number you first thought of." And he is right.

Again, suppose the next student says his answer is 10. The Scribe again reduces this number by 1/10, and announces that this student first thought of the number 9, which he did.

In the papyrus, the example the Scribe gives is the latter, and is the simplest one possible in integers. If he thinks of 2/3 of a number added to itself as being 1 2/3 of the number, and 2/3 of this, from his table, gives 1 1/9 of the number, then he can easily determine that 1/10 of this 1/9,

* Numbers in brackets refer to the notes at the end of the article.

which on subtraction leaves 1, so that his procedure in fact holds for any number at all. On this view, therefore, we do not need to suppose with Chace that the Scribe, so to speak, accidentally noticed that "9 is obtained by taking away its 1/10 from 10," and that "the solution does not seem to be complete." Indeed, the Scribe concludes definitely enough with, "Do it thus," or, as Chace also translates it, "The doing as it occurs," a proper injunction to the student at the end of his problem. Chace would further have no need to puzzle over the fact that "in no other problem are these words at the end of the solution," [2] nor to suppose that the problem is incomplete. We can now see the problem, in fact, as quite complete, succinct, and perfectly general, and in conformity with the format of the rest of the *Rhind Mathematical Papyrus*, in that the Scribe seldom explains his methods, being satisfied with showing by actual calculation ("The doing as it occurs") that his answer is correct.

We proceed now to Problem 29.

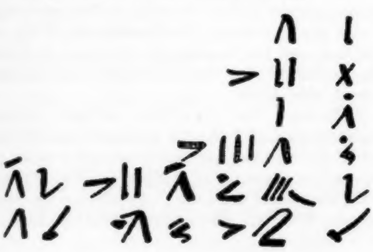


Figure 2: Problem 29

Chace gives the translation of this problem as (Vol. II, Plate 51):

1	10
1/4	2 1/2
1/10	1
Total	13 1/2
2/3	9
Total	22 1/2
1/3	7 1/2
Total	30
2/3	20 [3]
1/3	10

But (Vol. I, p. 70):

A quantity and its 2/3 are added together, and 1/3 of the sum is added; then 1/3 of this sum is taken and the result is 10. What is the quantity?

1	10
1/4	2 1/2
1/10	1
The quantity is	13 1/2
2/3	9
Total	22 1/2
1/3	7 1/2
Total	30
2/3	20 [3]
1/3	10

As in the preceding problem it may be supposed that our author first solved the problem as follows:

Assume 27

1	27
2/3	18
Total	45
1/3	15
Total	60
2/3	40 [3]
1/3	20

As many times as 20 must be multiplied to give 10, so many times 27 must be multiplied to give the required number. But at this point he seems to have changed the order of these numbers in his mind, and to have said, As many times as 20 must be multiplied to give 27, so many times 10 must be multiplied to give the required number.

\	1	20
	1/2	10
\	1/4	5
\	1/10	2
	Total 1 1/4 1/10	27

Therefore we must multiply 10 by 1 1/4 1/10.

I suggest that Problem 29 is another of the Scribe's "Think-of-a-Number" problems, like Problem 28, and, if this is so, we can then understand why he dispenses with explanatory matter, for it stands in the *Rhind Mathematical Papyrus* alongside Problem 28 practically line for line, being meant to be a further example of the same type. We observe that Chace has added a full statement of what he con-

siders the Scribe had in his mind, for the better understanding of his readers, to enable them to follow what might otherwise be a meaningless jumble of figures. As for Problem 28 then, we may suppose the Scribe to have proposed from his notes that his students:

"Think of a number. Add 2/3 of this number to itself. Add 1/3 of this result to itself. Take 1/3 of this and tell me your answer."

Suppose the first student announced his answer as 10 (the number in fact which was chosen also in Problem 28). The Scribe then simply adds 1/4 and 1/10 on to the number, getting 13 1/2, which he at once announces was the number first thought of. His working of these steps is shown in the papyrus, followed, as is his custom, by a check proof for verification. Chace has endeavored to show how the Scribe arrived at his 1/4+1/10, and in my view his method of doing this is probably the correct one, although for a slightly different reason. It may be supposed that our author said that:

"One-third of 1 1/3 times 1 2/3 of a quantity is the answer. Or, 1 1/3 times 1 2/3 of the quantity is 3 times the answer. Or, 4 times 1 2/3 of the quantity is 9 times the answer. Or, 4 times 5 times the quantity is 27 times the answer. That is, 20 times the quantity is 27 times the answer. Thus, the required quantity is 27 divided by 20 times the answer."

Then, divide 27 by 20.

Multipy 20 until 27 is reached.

	1	20
	1/4	5
	1/10	2
Total	1 1/4 1/10	27

Thus, whatever answer is given, the Scribe merely adds 1/4 and 1/10 to it, and obtains the number first thought of.

NOTES

1. From the Introduction to the *Rhind Mathematical Papyrus*, Chace, Vol. I, p. 49: "Accurate reckoning. The entrance into the knowledge of all existing things and all obscure secrets. This book was copied in the year 33, in the fourth month of the inundation season, under the majesty of the king of Upper and Lower Egypt A-user-Re endowed with life, in likeness to writings of old made in the time of the king of Upper and Lower Egypt, Ne-ma-et-Re. It is the scribe Ah-mose who copies this writing."
2. Chace continues: "Peet has suggested that, in copying, the scribe came to these words and unconsciously let his eye pass to the same words in the next problem, the statement of the next problem and the beginning of its solution being also omitted." On the evidence, I am unable to accept this view.
3. Following the Egyptian scribe's standard practice, to find 1/3 of a number, he first takes 2/3 of the number, probably from his table, then halves his answer. *C.f.* Gillings, "The Egyptian 2/3 Table for Fractions," *Australian Journal of Science*, Vol. 22, December, 1959, p. 247.

"One over . . ."

by William R. Ransom, Tufts University, Medford, Massachusetts

Editorial Note: At the start of the *Rhind Papyrus*, we find a table giving particular decompositions of fractions of the form $2/n$, for odd n from $n=3$ to $n=101$, as sums of unit fractions, or fractions with unit numerators. Since there are many different ways of expressing a fraction of the form $2/n$ as a sum of unit fractions, it is natural to wonder just what rule or rules the ancient Egyptians employed to obtain the particular decompositions listed in the *Rhind Papyrus*. There has been much speculation on this problem and the interested reader might consult the following: (1) Otto Neugebauer, *The Exact Sciences in Antiquity* (Princeton: Princeton University Press, 1952), pp. 74-77, (2) D. E. Smith, *History of Mathematics* (Boston: Ginn and Company, 1925), vol. 2, pp. 210-211, (3) B. L. Van der Waerden, *Science Awakening* (Groningen, Holland: P. Noordhoff Ltd., 1954), pp. 23-26.

Some one claims that when you take one-fourth, or one over anything, it is the

last one of the equal parts, because that is the one which assures you that the partition comes out even! But this remark would not apply to $1/\pi$ or to $1/\sqrt{2}$.

The "one over $\cdots$" fractions were important in ancient Egypt. For the sum $1/3+9/14+10/21$, the Egyptian Ahmes (about 1650 B.C.) would change to

$$1/3+(7+2)/14+(3+7)/21$$
$$=1/3+1/2+1/7+1/7+1/3,$$

(instead of, as we should, $(14+27+20)/42=61/42$), but that would give

$$1/2+2/3+2/7,$$

and that was not "one over $\cdots$" enough, and had to be changed to

$$1/2+(1/2+1/6)+(1/4+1/28)$$
$$=1+1/4+1/6+1/28.$$

For the "two overs," Ahmes had a table for all odd denominators, from 2/3 to 2/101. We do not know what system he used in preparing this table. Four of the entries were as follows:

$$2/3=1/2+1/6$$
$$2/43=1/22+1/946$$
$$2/95=1/60+1/380+1/570$$
$$2/99=1/90+1/110=1/50+1/4950.$$

To find a set of "one overs," or unit fractions, a multiplier M is tried. For a given odd number N take $2/N=2M/NM$, and if among the divisors of NM a set can be picked out whose sum is $2M$, the $2/N$ can be transformed as desired. For example, take 2/15, with $M=4$. We then have 8/60, and the divisors of 60 (we need not go beyond 8) are 1, 2, 3, 4, 5, 6, 10, $\cdots$. From these we can pick out four sets with sum equal to 8:

$$6+2, \qquad 5+3, \qquad 5+2+1, \qquad 4+3+1,$$

and from these our 2/15 or 8/60 becomes $1/10+1/30$, $1/12+1/20$, $1/12+1/30+1/60$, or $1/15+1/20+1/60$.

Besides $M=4$, for 2/15 one can use $M=2$, $M=3$, but not $M=5$, $M=6$, $M=7$, etc. For his 2/43, Ahmes might have used $M=6$ (but no smaller number), but his result corresponds to the use of $M=22$.

Two rules have been proposed by which Ahmes might have found a usable value of M directly, instead of by trial. But most of his results are not in accord with these rules, so we suppose he did not know them. He of course did not have the advantage of our present system of algebra.

One rule for $2/N$ is to use $M=(N+1)/2$; this gives $2/N=1/M+1/MN$. The other rule factors N, say $N=rs$, and takes $M=(r+s)/2$, which gives $2/N=1/rM+1/sM$. If N is a prime number, taking $r=N$ and $s=1$ gives the same result as the first rule.

The binary notation, for which computers have recently developed a wide interest, gives a new rule for choosing M. If 2^n is the largest power of 2 less than N, we have the inequality

$$2^n<N\leqq 2^{n+1}.$$

This shows that $2^{n+1}-N$ is a positive number less than 2^{n+1}, and so can be expressed as a sum of powers of 2 not greater than 2^n, each power being a divisor of $N\cdot 2^n$.

For example, take 2/43. The power of 2 next below 43 is 32. So we take $M=2^5$ and get $2/43=64/1376$. Now $2^6-43=21$. To put 21 into binary form, take half, and half again, etc. (rejecting the remainders), and write the powers of 2 under them, thus:

21	10	5	2	1
1	2	4	8	16

Taking the powers under the *odd* numbers, we obtain $1+4+16=21$. So we can express the 2/43 as

$$64/1376=(43+1+4+16)/(43\cdot 32)$$
$$=1/32+1/1376+1/344+1/86.$$

For spare-time practice with fractions, forming a table for the "two overs," or finding how many forms you can get for one of them, is a rather interesting game.

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Advanced Algebra, E. A. Maxwell (London, England: Cambridge University Press, 1960). Cloth, ix+311 pp., $2.75.

According to Mr. Maxwell this book "... ought to form a basis for most upper-school requirements below full Scholarship level, and, in places, beyond." There is no question as to the placement of this book in our own system of education. It definitely contains "college algebra" material and beyond. Examples of the chapter headings are "Complex Numbers" (Chapter IV), "Introduction to the General Polynomial" (Chapter V), "Partial Fractions" (Chapter X), "Inequalities" (Chapter XI), "Infinite Series" (Chapter XVI), "The Binomial Series" (Chapter XVII), "The Exponential Series" (Chapter XVIII), and "The Logarithmic Series" (Chapter XIX).

Mr. Maxwell emphasizes the logical structure of algebra quite well and the language used could be called "modern" in tone, although the book itself is traditional. In the second section of the first chapter, he discusses the function concept and follows this in the next two chapters with the problem of finding the zeros of linear and quadratic polynomials. Of special interest in Mr. Maxwell's discussion of linear polynomials is the section on the unique determination of a linear polynomial when two assigned values of the polynomial are given for two distinct values of the independent variable (page 12). Quoting from the book, "Suppose that the polynomial has values L, M for $x=p$, q, with $p\neq q$. ... Observe that the function

$$\frac{x-q}{p-q}$$

has values 1, 0 for $x=p$, q and the function

$$\frac{x-p}{q-p}$$

has values 0, 1 for $x=p$, q. Thus the expression

$$L\left(\frac{x-q}{p-q}\right)+M\left(\frac{x-p}{q-p}\right)$$

which is, on expansion, the linear polynomial

$$\left(\frac{L-M}{p-q}\right)x+\left(\frac{Mp-Lq}{p-q}\right),$$

takes the required values L, M for $x=p$, q." He then proves the uniqueness of this expression. This, of course, is developed without aid of graphical methods and, as noted above, done very early in the book. Although interesting, it might not be called commendable from a pedagogical point of view. This example rather well demonstrates the presentation throughout the rest of the book.

Mr. Maxwell does present permutations and combinations before presenting the binomial theorem. Although this is not unusual, many books do not do this. This is, of course, quite commendable since the presentation of the binomial theorem is thereby simplified. The section on probability in the chapter on permutations and combinations is shorter than in most comparable books; but no claim is made that probability, although interesting, need be taught at this stage.

The last chapter, titled "Elementary Properties of Determinants," is more extensive than is usually found in textbooks of this nature. The author calls the treatment "orthodox" but admits that he "... fixes attention on those parts of work which form a preparation for the important subject of 'linear algebra.'" His presentation, though "orthodox," is fundamentally good.

In general, this textbook is a sound and logical presentation of the area of elementary algebra. Though very traditional (the author says "fairly standard") it contains a wealth of information and *many* exercises. The cost of the book ($2.75) is an item that has merit in itself. Any student who completes this book will be prepared admirably for future mathematical work. No algebraic manipulation in the calculus will be beyond him and his work in matrix theory will have the necessary foundations.

On the negative side, the book is quite difficult and formidable. Without a large amount of help from the instructor, only the most able and mature student can handle the material. Although symbols are rather universal in nature, care must be taken with the exactly reversed placement, compared to use in the United States, of the decimal points and period for multiplication. Giving the book the true "foreign flavor" are those problems concerning the monetary system in England. Unless a person is quite familiar with this system the problems may have to be omitted. Some people claim that any coin problems should be omitted although probability naturally adapts well to "flipping pennies." On the other hand, whether an American penny or an English penny was flipped, the effect on the probability of heads is rather small.—*Edward H. Whitmore, San Francisco State College, San Francisco, California.*

Companion to School Mathematics, F. C. Boon (London, England: Longmans, Green, and Company, Ltd., 1960). Cloth, 302 pp., 30/0, approximately $4.50.

This book meets an urgent present-day need, for it provides an opportunity for able students to do individual study in mathematics. The material of the book is presented in an interesting manner and also provides substantial mathematics.

The book contains twenty-three relatively short chapters which include the following: "Historical Outline," "Euclid's Postulates," "Squaring the Circle," "An Introduction to Some Curves," "Pythagoras' Theorem," "Symbols and Conventions," "Nomenclature," "Symmetry," "Analogy," "Continuity," "Negative Magnitudes," "Complex Numbers," "Generalizations and Extensions," "Inequalities," "Induction," "Paradoxes and Fallacies."

Almost all of the chapters are rich with historical references. A pupil reading this book, or even only certain sections of it, cannot fail to realize that mathematics has been fashioned by the mind of man. He will also understand that mathematics grew from rather meager beginnings to an extensive body of systematized, or related, knowledge. Chapter I lists the names of about ninety mathematicians and gives a short biography of each. This serves to emphasize the fact that mathematics has been man-made during centuries of time. This thought is further emphasized by the practice of mentioning the mathematician's name and his specific contribution as mathematical topics are developed in subsequent chapters.

The contents of the book are admirably suited to give secondary pupils a rich insight into the nature of mathematics. Disgressions into history, etymology, paradoxes, and glimpses of the way ahead are both interesting and challenging. It should prove a valuable aid in encouraging pupils to start on basic and simple research. Perusal by high school pupils of these topics will supply new knowledge and provide excellent maintenance exercise.

The index of this book is limited. Names of mathematicians are listed, but not the mathematical topics. Discussion of the Three Classic Problems or the Conic Sections, for example, can be found only by paging through the text.

In the discussion of the conic sections, although the spheres inscribed in the cone are indicated in the drawing, no mention is made of them as Dandelin Spheres. In the discussion of Squaring the Circle, the value of pi is given to 150 places and the works of Sharp, Machin, Rutherford, and Shank are cited, but no mention is made of the correction of Shank's calculation made a decade or so ago. Nor is there reference to the 1950 calculation of pi to 2035 decimal places by George W. Reitwiesner, Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland. Indeed, this points to the fact that the 1960 edition of the book differs from the 1924 edition only in the inclusion of a foreword and a new bibliography. The new bibliography consists of only seventeen books, the majority of which were published prior to 1950.

The paragraphs dealing with mechanical construction of many important curves will be helpful to students wishing to build mathematics teaching aids.

Only a little space is devoted to a discussion of the metric system, a topic whose study is rapidly gaining in popularity. No reference is made to concepts of the so-called modern mathematics such as sets, fields, groups, graphing of inequalities, numeration systems in bases other than ten, or properties of a number system. The topic of non-Euclidean geometry, however, is given a fair amount of space.

At the present time we are interested, in the United States, in moving away from the compartmentalized method of presenting secondary mathematics to pupils. The following paragraph taken from Chapter X, page 155, illustrates the author's idea of using a historical approach to develop this sought-after sense of unity in the structure of mathematics: "It is no longer considered necessary in elementary teaching to separate one branch of mathematics from another—algebra is admitted into arithmetic and geometry; graphs are used everywhere. But it is perhaps insufficiently recognized how in the history of mathematics the knowledge of one branch has helped the development of another and how an illustration from one branch may illuminate a piece of work in another."—*Lee E. Boyer, Department of Public Instruction, Harrisburg, Pennsylvania.*

Geometry, Charles F. Brumfiel, Robert E. Eicholz, and Merrill E. Shanks (Reading, Massachusetts: Addison Wesley, Inc., 1960). Cloth, xi+288 pp., $4.75.

One of the many unique features of this high school geometry book is its second chapter which is devoted to a study of elementary logic. Nonmathematical statements as well as statements in the form of equations provide a variety of examples of negation and compound statements. An investigation of the meaning of statements containing the connectives "and," "or," and "if . . . then . . . " culminates in the defining truth tables. There is an abundance of exercises which lead to an understanding of the use of truth tables. In using this book as a text for two high school geometry classes, the reviewer found the students readily convinced themselves by truth tables that an "if . . . then" statement and its contrapositive are logically equivalent. This equivalence is utilized in indirect proofs throughout the book.

The authors state in the preface that their main objective in writing the text was to produce "a mathematically adequate and yet elementary treatment of plane geometry in the spirit of Euclid." To remedy the logical gaps in Euclid, their treatment makes wide departures from traditional texts.

The undefined terms, point, straight line, between, and congruence, are clearly stated as such. The definitions utilize the concept of set, although the symbolic set notation is not used. Indeed, geometric figures are considered to be sets of points. A careful distinction is made between the abstract notion of collections of points which are the primary concern and a drawing which merely serves as an illustration.

The postulates are eighteen in number and are introduced as they are needed. There is a group of five postulates having to do with the betweenness relation. These five postulates together with three existence or so-called incidence postulates are the foundation for achieving a rigor impossible in most high school geometry texts. Proofs of constructions and proofs that introduce supplementary sets of points rely heavily on the betweenness and incidence postulates.

The $>$ and $<$ relations for line segments are defined in terms of congruence and betweenness. This allows a comparison of segments before the measurement of segments is defined. In an analogous development for angles, the relations of congruence, $>$, and $<$ for angles are not defined in terms of the measurement of angles. This is quite different from the traditional approach, but we need to keep in mind that this is a geometry dealing with abstractions rather than with physical objects that can be measured with a ruler. An immediate advantage of such a treatment is the unmistakable distinction between establishing the truth of a statement by deductive reasoning from the postulates, definitions, and proved theorems and verifying by measurement that the statement is true for selected illustrative drawings.

Algebra enters the logical structure via the study of similar polygons involving ratio and proportion and provides experience in proofs of an algebraic nature.

The notion of limits is introduced in connection with the study of circles and plays an important part in the definitions of circumference and area of circles.

Because measurement of angles is defined in terms of the length of the arc subtended by a central angle of a circle, it is natural to define radian measure along with degree measure of angles. Thus students are introduced to both of these units of angle measurement prior to their use in trigonometry.

Many of the theorems which are not essential to subsequent development appear as exercises for the student to prove. Solid geometry is treated in a single chapter with the necessary additional postulates and the major theorems listed. Very few proofs of the solid geometry theorems are given in the text. There is a carefully developed introduction to plane co-ordinate geometry, including a proof of the distance formula.

Precision in the wording of this text often necessitates a loss of simplicity. In the opinion of the reviewer, set notation (including set operations) would provide a conciseness not obtainable in the worded statements. Students always approve of short cuts and would quickly see the advantage of such a shorthand.

At first glance, it might appear pedagogically unwise to hand "Euclid made precise and rigorous" to high school students on a silver platter, so to speak. But is it better to provide the student with a sound mathematical structure and let him gain mathematical experience by discovering and proving theorems within that structure or to require him to use an inadequate structure in building a synthetic geometry?

In conclusion, this textbook combines the spirit and contents of modern mathematics with the time-honored body of knowledge of geometry. Any teacher looking for a fresh approach to geometry will be interested in this book and will find it an asset in increasing his own understanding of geometry.—*Eleanor V. Dean, University School, Florida State University, Tallahassee, Florida.*

Mathematics in Action, O. G. Sutton (New York: Harper and Brothers, 1960). Paper, xvi+236 pp., $1.45.

This is one example where the title is consistent with the content of the work. The style is precise and accurate with an even flow that carries the reader along with a minimum of abstractions as the author reveals the role of mathematics in applied problems and especially the theories of the physical universe. James R. Newman aptly states in the foreword, "It is a wonderfully exciting tour through the workshops of the mathematical physicist. . . ."

The work belongs in the category of books popularizing mathematics, since it deals with the triumph of mathematics in the physical sciences and man's search into the nature of the universe. The procedure for the mathematical solution of a problem in physics reveals the use of mathematical models and the scientific method of researchers.

There is first a brief discussion of the role of the mathematician in the study of the theories of the physical universe. This is followed by an equally brief statement of mathematics as a "tool of the trade." One unfamiliar with the calculus and some classical theory in physics might find it difficult to follow the discussion of topics such as the difference between the derivative and the differential, Laplacian equations, field theory in physics, mathematical waves, ballistics, Fourier series, fluid motion, and other classical topics. However, the author has done well in his brief, accurate, and concise exposition. It is abundantly clear that the body of knowledge about the physical universe is the result of a safari into applied mathematics.

The chapter on statistics is significant and timely in view of the increased demand for statistical analysis in so many areas of learning and the false conclusions presented by workers not familiar with the mathematics of statistics. The discussion on sampling should cause those to pause and think who treat all samples as if

they fit the Gaussian curve and represent the parent population. This is particularly significant since statistics has become so important in the behavioral sciences.

This reviewer heartily recommends the work to all libraries, to sciences classes for careful supplementary reading, and to the layman with some intellectual curiosity in the area of science. —*H. Glenn Ayre, Western Illinois University.*

The Modern Aspect of Mathematics, Lucienne Félix, translated by Julius H. Hlavaty and Francille H. Hlavaty (New York: Basic Books, Inc., 1960). Cloth, xiii+194 pp., $5.00.

Irrational numbers, divergent series, and continuous functions with no derivative for any values of the variable have caused "scandals" in the past. Mlle. Félix recounts many of these scandals and then states that today's scandal is on the level of teaching. "No teacher of mathematics at the secondary level should be ignorant of what his students will be studying the following year in preparing for teaching or engineering." The author strives to stimulate interest in the modern aspects of mathematics, hoping thereby to reduce the magnitude of this scandal.

The Modern Aspect of Mathematics is historical to a rather large degree. Displaying considerable scholarship, the author recounts historical developments, from Hugens' treatise on locks and Lord Kelvin's on ether, to the electromagnetic theory of Maxwell who interpreted the waves of Fresnel and to Planck and the "quantum of action." The emphasis is on the stimulus received from natural science which, Picard stated, "puts order, at least tentative order, into nature."

Mathematical logic and the theory of sets, representing the substructure of mathematics, are developed. Some of the examples represent the Bourbaki treatment of sets, the Bourbaki being a group of French mathematicians publishing a treatise, the *Elements of Mathematics*. To date, over twenty volumes in this treatise have been completed.

General algebra and general topology, however, represent two pillars supporting the whole structure. Groups, rings, fields, and vector spaces are defined. Groups are illustrated by a model whereby symmetries and rotations leave an equilateral triangle invariant.

After citing certain major theories in mathematics in a "rapid review," the author explains how mathematical extensions could occur: (a) through extension of a theory, e.g., natural numbers to the rational numbers; (b) transforming the definition of the operation by enlarging it, as did Lebesgue in the work on the integral.

For many teachers of mathematics, the closing chapter, "The Pedagogic Point of View," is especially worth reading. Those who oppose all curriculum changes are advised that "every change of residence is to some extent a catastrophe for the person involved."

Mlle. Félix's book is designed to portray the history and the modern aspects of mathematics as part of a broad cultural approach. The book is not a textbook, but it does introduce elements of modern mathematics in a readable manner. It will stimulate interest in these topics. It is " . . . up to each of us to pursue the solution. It would be denying our mission not to profit from the breath of fresh air that is offered to us, not to find again the enthusiasm of youth."

Appendix I, "An Example of the Applications of Logic," and Appendix II, "An Example of the Use of Formalism," contain material that may be used in the classroom. This material is technical, but not so technical as the section, "Composite Algebraico-Topological Structures," pp. 120–127. Although this section is interesting, it is of interest primarily to a professional mathematician. I believe the proof on page 126, that the p-adic norm is not Archimedian, is not correct. However,

$$|a+a+a \cdot \cdot +a|_p \leqq |a|_p,$$

which suffices for the proof.

Mlle. Félix's association with the Bourbaki places her in a position to know the modern trends, understand their implications, and appreciate the need for their adoption in our curriculum. This book will help each reader place himself in a similar position of strength.—*J. Raymond Silva, Pelham High School, Pelham, New York.*

Fun With Mathematics, William H. Glenn and Donovan A. Johnson (St. Louis: Webster Publishing Company, 1960). Paper, 43 pp., 56¢.

Number Patterns, William H. Glenn and Donovan A. Johnson (St. Louis: Webster Publishing Company, 1960). Paper, 47 pp., 56¢.

Sets, Sentences, and Operations, Donovan A. Johnson and William H. Glenn (St. Louis: Webster Publishing Company, 1960). Paper, 63 pp., 64¢.

Topology, the Rubber-Sheet Geometry, Donovan A. Johnson and William H. Glenn (St. Louis: Webster Publishing Company, 1960). Paper, 40 pp., 52¢.

Understanding Numeration Systems, Donovan A. Johnson and William H. Glenn (St. Louis: Webster Publishing Company, 1960). Paper, 56 pp., 60¢.

Any mathematics teacher who, in the past, has been uncertain as to a proper response to the oft-asked question, "What can I do for extra credit?" needs hesitate no longer. With a set of the above-listed booklets, he can encourage students to explore mathematics on their own. This reviewer thinks that these colorful and cleverly designed booklets, which have a modern approach to mathematics, are excellent. The answer sheets to the sets of exercises can be removed before the booklets are given to the students. Because this writer feels that this series, of which there will be twelve by late 1960, will be used so extensively, she, in hopes of saving time for many, will mention some errors which appear in the answers.

The main idea expressed in *Understanding*

Numeration Systems is that new insights into our base ten system can be gained from an understanding of numeration systems with other bases. The four fundamental operations are used with bases of five, twelve, and two. The use of nomographs and Napier's bones is described. An explanation is given of how the binary numeration system is used by electronic computers. Recreational uses of the base two are given in describing games of nim and of age guessing. Russian or peasant multiplication could have been included in this section.

The subscript five was omitted from the answer to exercise 2 in Set 6, as was the subscript twelve from the answer to exercise 40 on page 50. The answers to exercises 1(d) and 2(d) in Set 16 were omitted. On page 30, when trying to express 5/9 as a base 12 rational, the reader is advised to "express 5/9 as a base 10 fraction whose denominator is a multiple of 12." The word "multiple" should be replaced by the word "power," since grouping is done by dozens.

Even though the first sentence in *Fun With Mathematics* contains the common grammatical error of using "as" instead of "so" with a negative, this booklet is a good one for students in junior high mathematics and/or first year algebra. In presenting tricks and puzzles, the authors have gone beyond the usual presentations found in recreational mathematics books to help the student explore the mathematics of the how and the why. One of these tricks will add to the student's understanding of the place-value concept used in our numeration system. Problems involving cards, dice, and dominoes are given along with explanations which should help students become more skillful in dealing with mathematical situations.

Students are asked to calculate the date of Easter Sunday for the next year and also to determine the day of the week on which they were born. Under the heading "The Impossible Isn't Too Hard," a clever way to trisect an angle is given. The restriction of instruments to compass and ruler is removed, and a clock is used.

Exercise 3 in Set 3 requires the use of the formula $\frac{x(x+1)}{2}$ for finding the sum of the first x consecutive integers. First-year algebra students probably would not know this formula. To have the answer to the exercise in Set 5 consistent with the letters used in the exercise, it should read "21-*e*."

The booklet *Number Patterns* emphasizes the idea that mathematics is often considered to be a study of patterns. Algebra is used to explain many of the well-known puzzles which appear in popular magazines. How many of the teachers who show their students how to "cast out nines" can really explain this process? The authors state, "Although it is interesting to note the patterns displayed by the answers obtained in the problems, the real challenge lies in proving the 'why' of the problem."

A teacher of arithmetic or general mathematics could develop many of these patterns with his students. What a lot of fun they could have without realizing they were getting large doses of practice using number facts.

In applying number patterns to interest problems, an error was made in the answer to exercise 3 in Set 10. An interest table is to be used to find the amount of one dollar invested at simple interest at 2%, 4%, and 6%, for 60 years. The suggestion is made in the answer to use the values in the table for $n=50$ and $n=10$. Each of the answers is one dollar too much, because by using the formula $A_{60}=A_{50}+A_{10}$, the principal has been added an extra time. At the bottom of the table is the correct formula: $A_{60}=1+.02n$. In the same exercise, the use of the interest table to find the amount of one dollar invested at compound interest at 2% for 60 years is a good illustration of the use of exponents.

Other number patterns are presented in developing magic squares and in showing formulas used through the years for computing the value of π.

The booklet *Topology, the Rubber-Sheet Geometry* is an introduction to the strange world in which one deals with the properties of position that are not affected by changes in size and shape. Trying to solve the puzzle of the Koenigsberg bridge leads to experiments with networks. Moebius strips, the four-color problem, Klein's bottle, knots, and the application of Euler's formula to polyhedrons are discussed. The authors state that topology is becoming a valuable tool in our complex space age.

The booklet *Sets, Sentences, and Operations* explores the essential meaning of the set idea, which is shown in operation in geometry, algebra, with work on equations and inequalities, in graphing of solution sets, and in logic. The ideas of set concepts are stressed as being more important than set notation. This reviewer believes this booklet does a better job than similar publications in convincing students (as well as doubting teachers) that there is value for high school students in understanding set ideas. Boys should enjoy the exercise in which they are required to use Venn diagrams to determine the American League teams that played on a certain day if the probable winners picked by three sports writers are known. In the answer to exercise 5(a) in Set 1, 99 should not be included.

This reviewer has been puzzled about how to get students to read mathematics books which are in the school library and about how to have them report if they do read such books. These booklets help with both problems. Students might have time to read some of these, yet would not have time to read a book. There is enough material in each of these to discourage the student who is merely seeking activity to boost a sagging grade average. The solutions to the sets of exercises could be sufficient for a report. Students of this writer have enjoyed this fine new series.—*Mary Reed, Benton Harbor High School, Benton Harbor, Michigan.*

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

The man in motion

by Robert J. Mills, Shaler High School, Glenshaw, Pennsylvania

A few years ago, George Hallas, coach of the Chicago Bears professional football team, won a championship by baffling his opponents with a formation using the "man in motion." In coaching circles this was hailed as an innovation, but any beginning teacher could have told the coaching fraternity there was nothing new to being baffled by a "man in motion." I would like to present an approach to motion problems with which I have had some success.

I believe one stumbling block to success on the part of the beginning algebra student is the unusual position in which he finds himself when confronted with his first motion problem. The student suddenly finds himself with too much information. Rather than having to "dig" for facts to write his equation, he figuratively drowns in rates of speed, starting times, directions of motion, and time in motion. Since this is an unusual position in which to find oneself in algebra, why not use an unorthodox approach to the problem? Present the student with a line of reasoning to follow so that he can at least get a start on solving this type of problem. This line of reasoning consists of two concepts: (a) if you can find the rates of speed and time, you can find the distance; (b) if you can find the distance, you can write the equation to solve the problem.

The first concept is developed in the traditional manner by use of the distance formula and the "box" or "scoreboard." (See Fig. 1.)

	Rate	Time	Distance
A			
B			

Figure 1

Suggest now that the student fill the rate and time blocks first. This is done for three reasons: (a) if you can express the rate and time, simple multiplication will give you an expression of the distance; (b) if you solve for distance, you will avoid algebraic fractions which develop if you solve for rate or time; (c) the instructor can vary the rates or starting times to teach the students to write algebraic expressions of either. The problems should be varied enough to prevent the students from blindly memorizing a solution to all problems of this type.

Having once arrived at an expression of the distance each object has travelled, the second concept can then be developed. A motion problem may be placed in one of four categories: approach, separation, pursuit, and round trip. Each may be taken separately and analyzed in such a way as to develop an equation.

In approach problems, two objects usually start toward each other from a given

distance apart. They continue until they meet or pass. (See Fig. 2.)

Figure 2

Therefore, regardless of their individual rates or times, the distance "A" travels (D_1) added to the distance "B" travels (D_2) equals the total distance (D_t). This equation may be written as $D_1+D_2=D_t$.

In separation problems, two objects usually start at a common point and move in opposite directions until they are a given distance apart. Again, regardless of their rates or times, the distance "A" travels (D_1) added to the distance "B" travels (D_2) equals the total distance given (D_t). (See Fig. 3.)

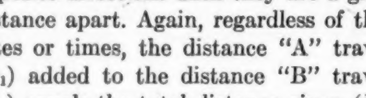


Figure 3

Thus the solution of these two types of problems reduces to one line of reasoning, that of adding to get the total distance.

In pursuit problems, one object usually leaves a common point before the other, and then the chase begins. But when one object overtakes the other, the total distance travelled by one equals that of the other. Thus the head start added to the distance travelled by "A" after "B" starts will equal that travelled by "B." (See Fig. 4.) The equation is "Head start" $+D_1=D_2$

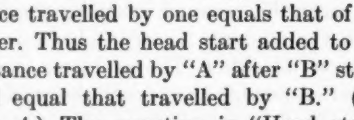
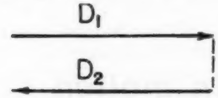


Figure 4

In round-trip problems, one object usually travels away from a given point and returns at a different rate of speed within a given time. Here there is an opportunity to write the equation two ways: (1) the distance going added to the distance returning will equal total distance, or (2) distance out equals distance back. (See Fig. 5.)



$$D_1+D_2=D_t$$

$$D_1=D_2$$

Figure 5

The second is preferred because often the total distance is not given in round-trip problems. Also in this manner, pursuit and round-trip problems may be solved by the concept of distance equals distance.

In summary, if the student can write an algebraic expression for the distance each object travels, it is possible to write a solution equation by adding these distances or by setting them equal.

This method of approach may be criticized as mechanical, and one may ask, "Where is the element of individual discovery?" There is little chance that beginning algebra students have not had contact with the distance formula. Therefore, one can concentrate on helping the students develop the concepts of "equal distances" and "total distance." This may be done by placing students in positions on opposite sides of the room and setting up the conditions of the given problem, such as starting times, rates of speed, and stopping times. The students could then be questioned on each factor of the formula. What do they know about the speed? about the time? about the distance? Students often know from the given conditions of the problem that the times, rates, and distances are unequal; but they overlook the important concept of the total distance both objects move. By repeated questioning about the distance and by giving the room an exact width, some students will still see only that the objects move unequal distances. But a few will see that they also move the total width of the room. This concept of total distance is the important point to work

toward. For separation, pursuit, and round-trip problems, the process may be repeated; and the students will soon learn what to look for in each problem. This helps to overcome the initial stumbling block mentioned, that of too much information for the student to analyze. The student will also realize that approach and separation problems work on one principle, while pursuit and round-trip problems work on another.

Once a solid foundation has been developed by working to solve for the distance, the door is open for expansion into solutions for time or rate. This leads into fractional equations, and gives the teacher an opportunity to let students develop solutions of their own. This may be done by presenting problems where the total distance is not given, and the student is forced to solve for the rate and time.

In presenting problems involving different starting times, one must be careful. Beginning students, if told that "A" starts at 7:00 A.M. and "B" at 9:00 A.M., will invariably express "A" 's starting time as x while "B" 's becomes $x+2$. One must develop the idea that it is the total time in motion that is important. Thus "A" 's time should be expressed as $x+2$, while "B" 's is x.

When Coach Hallas revived the T formation and the "man in motion," other coaches immediately adopted it as their basic formation. The more inventive coaches soon introduced variations such as the split T, winged T, and slot T. Just as the coaches used the T formation as a starting point, so will some students use the above approach to motion problems for the same purpose. Your more inquisitive students will also invent or find variations of attack. That is the purpose of this method; it is to serve as a starting point for student and teacher, not as a mechanical method of solving motion problems.

A mathematics seminar

by Frank Anderson, Phoenix Union High Schools and Phoenix College System, Phoenix, Arizona

Among the ever-present problems in teaching mathematics are those of how to teach high school students more mathematics than they cover in their regular classes and how to create and hold their interest. A mathematics seminar group at Phoenix Camelback High School has been a partial solution to these problems. This study field trip group is limited to those students who have completed three years of mathematics and are currently enrolled in their fourth year.

An enthusiastic mathematics teacher gives a regular lesson on modern algebra to this group one afternoon each week. Among the topics covered are truth tables with their applications for compound statements and switching circuits; set theory and Venn diagrams with applications in partitions, counting, permutations, and the binomial theorem; and other number bases with particular study of the binary system because of its applications in computers. Each seminar member must attend these weekly meetings and prepare the regular assignments for them.

These weekly meetings are augmented with lecturers from industry who present the applied side of a mathematics topic. For example, the educational director of a General Electric computer division gave a three-lesson series on the computer and the binary algebra behind it. He demonstrated the binary computer by using members as tubes of the machine. These

human tubes, performing the operations of the machine, learned the routine necessary for actual operation of a computer process. This was followed by a field trip to see the computer in action.

Field trips by this group are taken on week ends or school holidays. Transportation is by private car. Every car on the trip must have an adult in it, and parents take turns furnishing transportation. One of the best outcomes of the program is the enthusiastic support of parents. An oft-repeated comment is that they are happy for the opportunity to go with their teenagers and to meet their children's contemporaries.

Large companies are very co-operative about visits. We contact the head of the company in our area about the group's interest, and usually the company is enthusiastic about arranging our visit. Three large international mining companies have taken as many as sixty students at a time through their open-pit operations, and through their milling and smelting plants. The Arizona Portland Cement Company showed its whole process from mining to delivery; and the students found the packaging most fascinating because of the automatic filling, weighing, and sealing of the bags. The local power plant has volunteered to show its new steam-generator plant which is operated by automation and which is large enough to furnish power to a city of 500,000. Students are amazed to find three trained engineers in a control room completely operating the huge installation by watching the dials and the small television sets trained on gauges two blocks away or seven stories up. Observatories, steel foundries, state highway departments, and Frank Lloyd Wright's architectural school are among other places giving their services. Our local professional engineers' association has been a wonderful source of ideas and company contacts.

Two field trips have been the high lights of each year's program. One is a Saturday-Sunday trip to the top of a 9,000-foot mountain to an Air Force radar station on the southern defense zone. Here all planes within a 250-mile radius are spotted on the huge plastic screen. Students may have seen this on television or movies, but nothing replaces the thrill of breathing over the shoulder of a man as he follows a 707 jet across the screen at 700 m.p.h., or of reading the radar and picking up the pip as a strange craft enters the zone. Lodging and food are arranged ahead of time by a student committee under adult guidance.

The grand tour of the year is a 600-mile trip of two nights and two days duration. The first night is spent with the U.S. Naval Observatory and the Lowell Observatory. The next day we visit the Glenn Canyon Dam site. The Department of the Interior really makes the visit interesting with a trip through the two-mile tunnel down 700 feet to the bedrock where the cement is being poured and a walk across the famous 700-foot suspension bridge hanging 650 feet above the river. An excellent color film on the preliminary planning and first two years of work on the dam is available free. Our homeward trip includes a sight-seeing tour of the Grand Canyon.

Remember, these trips are available to those mathematics students who attend the regular after-school mathematics seminars and who do the required assignments. Field trips are to let the students see the applications and also include some fun in learning. While careful planning must be done, it is a wonderful way to get the teachers, students, and parents working together.

This mathematics seminar group has proved so successful that some method of limiting its size or splitting its membership will have to be arranged. We feel it is one of the important factors in our high percentage of students—three out of four—electing the optional tenth- and eleventh-grade mathematics courses. We recommend the seminar for its many valuable side effects as well as for its major purpose.

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Committees and Representatives (1960-1961)

At their meeting in December, 1959 and April, 1960, the Board of Directors of NCTM made a careful study of the committee structure of the National Council of Teachers of Mathematics. The chief purposes of this reorganization were:

(1) To provide to the Board of Directors more time and background information as they consider basic policies and action proposals. This is obtained by having subcommittee reports analyzed and synthesized by the major committees prior to submission to the Board.

(2) To provide both effective action and thoughtful search for new projects by defining committee tasks more clearly, eliminating overlapping, and extending the financial support given to them. Several important committees have had broad charges and made extensive recommendations. It is hoped to encourage continued thought for broad and sweeping plans while also providing implementation for them by defining more sharply the tasks of major committees and providing ad hoc committees to execute projects which they conceive.

This study resulted in a reorganization of many committees into five major committees, namely,

1. The Executive Committee
2. Professional Standards Committee
3. Professional Relations Committee
4. Publications Committee
5. Plans and Proposals Committee.

The definitions of the tasks of the following committees and the assignment of *ad hoc* committees to them or to carry out their recommendations is still under consideration by the appropriate major committees: membership, teacher education and certification, relations with industry, international relations, and television. It should be clear that these committees represent important NCTM interests and have made extensive reports which it is hoped can now be further implemented.

The report of appointments for 1960–1961 is as follows:

EXECUTIVE COMMITTEE

Philip Peak, Bloomington, Ind. (1961)
Henry Van Engen, Madison, Wis. (1961)

SECRETARY OF THE BOARD

Houston T. Karnes, Baton Rouge, La.

PUBLICATIONS COMMITTEE

Henry Swain, Winnetka, Ill. (1961), Chairman
Donovan A. Johnson, Minneapolis, Minn. (1963)
Paul Johnson, Los Angeles, Calif. (1963)
Ben A. Sueltz, Cortland, N.Y. (1962)
Henry Van Engen, Madison, Wis. (1962)

Ex officio nonvoting members:

Myrl H. Ahrendt, Washington, D.C., Executive Secretary
Robert E. Pingry, Editor, THE MATHEMATICS TEACHER
E. Glenadine Gibb, Editor, *The Arithmetic Teacher*
W. Warwick Sawyer, Editor, *The Mathematics Student Journal*

Plans and Proposals Committee

Robert E. K. Rourke, Kent, Conn. (1962), Chairman
Roy Dubisch, Fresno, Calif. (1962)
William Matson, Seattle, Wash. (1963)
Henry Van Engen, Madison, Wis. (1961)
Edwin E. Moise, Cambridge, Mass. (1961)

Professional Status and Standards Committee

John R. Mayor, Washington, D.C. (1962), Chairman
W. T. Guy, Jr., Austin, Texas (1962)
Frank B. Allen, La Grange, Ill. (1963)
Henry Syer, Kent, Conn. (1961)
Mildred Keiffer, Cincinnati, Ohio (1963)

Professional Relations Committee

Irvin Brune, Cedar Falls, Iowa (1962), Chairman
J. Houston Banks, Nashville, Tenn. (1962)
Joseph F. Senta, St. Paul, Minn. (1963)
Veryl Schult, Washington, D.C. (1963)
Howard Fehr, New York, N.Y. (1961)

Nominations and Elections Committee, 1961

Oscar Schaaf, Eugene, Ore., Chairman
Harold P. Fawcett, Columbus, Ohio
W. Eugene Ferguson, Newton, Mass.
Lenore John, Chicago, Ill.
Houston T. Karnes, Baton Rouge, La.
W. C. Lowry, Charlottesville, Va.
Ida B. Puett, Atlanta, Ga.
Max Sobel, Fair Lawn, N.J.
Lottchen Hunter, Wichita, Kan.

Nominations and Elections Committee, 1962

W. Eugene Ferguson, Newton, Mass., Chairman
Mike Donahoe, Carmel, Calif.
Harold P. Fawcett, Columbus, Ohio
Agnes Herbert, Baltimore, Md.
Lottchen Hunter, Wichita, Kan.
Lenore John, Chicago, Ill.
W. C. Lowry, Charlottesville, Va.
Irene Sauble, Detroit, Mich.
Oscar Schaaf, Eugene, Ore.
Max Sobel, Fair Lawn, N.J.

Place of Meeting Committee

Eugene Nichols, Tallahassee, Fla. (1961), Chairman
Mabel Baker, Pittsburgh, Penn. (1961)
James Nudelman, Cupertina, Calif. (1961)
Ella S. Porter, Houston, Texas (1962)
Lauren Woodby, Mt. Pleasant, Mich. (1962)
Myrl H. Ahrendt, ex officio, Washington, D.C.

Committee on Financial Policies, Budget, and Auditing

H. Vernon Price, Iowa City, Iowa (1961), Chairman
Burton W. Jones, Boulder, Col. (1963)
Bruce Meserve, Montclair, N.J. (1962)
Myrl H. Ahrendt, ex officio, Washington, D.C.

Reporting Elections, 1961

Oscar Schaaf, Eugene, Ore., Chairman
Phillip S. Jones, Ann Arbor, Mich.
Myrl H. Ahrendt, Washington, D.C.

Committee on Affiliated Groups

Eugene Smith, Wilmington, Del. (1961), Chairman
Mary Rogers, Westfield, N.J. (1962)
Virginia Pratt, Omaha, Neb. (1961)
Kenneth Skeen, Oxnard, Calif. (1961)
Annie John Williams, Durham, N.C. (1962)
Adeline Riefling, St. Louis, Mo. (1961)
Keene C. Van Orden, San Angelo, Texas (1963)
Catherine A. V. Lyons, Pittsburgh, Penn. (1961)

Research Committee

John Kinsella, New York, N. Y. (1962), Chairman
Kenneth Brown, Washington, D.C. (1961)
Howard F. Fehr, New York, N.Y. (1961)
Sheldon Myers, Princeton, N.J. (1962)
J. Fred Weaver, Boston, Mass. (1961)

Editorial Board of The Arithmetic Teacher

E. Glenadine Gibb, Cedar Falls, Iowa (1963) Editor
E. W. Hamilton, Cedar Falls, Iowa
Edwina Deans, Arlington, Va.
Clarence Ethel Hardgrove, DeKalb, Ill.
J. Fred Weaver, Boston, Mass.
Marguerite Brydegaard, San Diego, Calif.
John R. Clark, New Hope, Penn.
Vincent J. Glennon, Syracuse, N.Y.

Editorial Board of The Mathematics Teacher

Robert E. Pingry, Urbana, Ill. (1962), Editor
Jackson B. Adkins, Exeter, N.H.
Mildred Keiffer, Cincinnati, Ohio
Daniel B. Lloyd, Washington, D.C.
Z. L. Loflin, Lafayette, La.
Ernest Ranucci, Union, N.J.

Supplementary Publications Committee

Burton W. Jones, Boulder, Colorado (1961), Chairman
Marguerite Brydegaard, San Diego, Calif. (1962)
Kenneth Henderson, Urbana, Ill. (1962)
Margaret Joseph, Shorewood, Wis. (1962)
Jesse Osborn, St. Louis, Mo. (1961)
Helen Schneider, Oak Park, Ill. (1961)
H. C. Trimble, Cedar Falls, Iowa (1961)

Yearbook Planning Committee

Bruce Meserve, Montclair, N.J. (1961), Chairman
J. Houston Banks, Nashville, Tenn. (1962)
Ben A. Sueltz, Cortland, N.Y. (1963)

Editorial Committee, 26th Yearbook (Evaluation)

Donovan A. Johnson, Minneapolis, Minn., Chairman
Robert Fouch, Chicago, Ill.
E. Glenadine Gibb, Cedar Falls, Iowa
Max Sobel, Fair Lawn, N.J.
Ben A. Sueltz, Cortland, N.Y.
J. Fred Weaver, Boston, Mass.

Editorial Committee, 27th Yearbook (Talented)

Julius Hlavaty, New York, N.Y., Chairman
Albert Blank, New York, N.Y.
Joseph L. Payne, Ann Arbor, Mich.
Richard Pieters, Andover, Mass.
Harry Ruderman, New York, N.Y.
Henry Syer, Kent, Conn.
Henry Van Engen, Madison, Wis.

Evaluation of Experimental Programs Committee

Philip Peak, Bloomington, Ind. (1962), Chairman
Alice M. Hach, Ann Arbor, Mich. (1961)
Kenneth B. Henderson, Urbana, Ill. (1963)
Z. L. Loflin, Lafayette, La. (1962)
Myron Rosskopf, New York, N.Y. (1963)

National Council Representatives

AAAS Co-operative Committee on Science and Mathematics
Bruce Meserve, Montclair, N.J. (1962)
Educational Advisory Committee to Science Service
Veryl Schult, Washington, D.C. (1962)
Helen Cooper, Bethesda, Md. (1961)
Conference Board of the Mathematical Sciences
Burton W. Jones, Boulder, Col. (1961)
John R. Mayor, Washington, D.C. (1962)
U.S. Commission on Mathematics Instruction
Phillip S. Jones, Ann Arbor, Mich. (1961)
E. H. C. Hildebrandt, Evanston, Ill. (1963)
AAAS Council
James Zant, Stillwater, Oklahoma (1962)
Veryl Schult, Washington, D.C. (1961)
Committee on Co-operation in Teacher Education
Veryl Schult, Washington, D.C.
TEPS Meeting
Clifford Bell, Los Angeles, Calif.
M. Vere DeVault, Austin, Texas
Alice M. Hach, Ann Arbor, Mich.
William Matson, Seattle, Wash.
C. Richard Purdy, Hayward, Calif.
Henry Syer, Kent, Conn.
NEA Delegate Assembly
Irvin Brune, Cedar Falls, Iowa
Meetings:
NEA Joint Meeting, 1960, *Los Angeles, June* 29
Local Arrangements Chairman: Marian Cliffe Herrick
Program: Clifford Bell
Summer, 1960, Salt Lake City, Utah, August 2
Local Arrangements Chairman: Eva Crangle, Salt Lake City, Utah
Program: Phillip S. Jones, Ann Arbor, Mich., and Mildred B. Cole, Aurora, Ill.
Christmas, 1960, Tempe, Ariz., December 27
Local Arrangements Chairman: Lehi Smith, Tempe, Ariz.
Program: Mildred B. Cole, Aurora, Ill.
Spring, 1961, Chicago, Ill., April 5
Local Arrangements Chairman: Hobart Sistler, Berwyn, Ill.
Program: Cifford Bell, Los Angeles, Calif
Mildred B. Cole, Aurora, Ill.

Call for candidates

The Committee on Nominations and Elections is making plans for the 1962 election. Officers to be elected are: president, vice-president for senior high school, vice-president for elementary school, and three directors.

The Committee invites your suggestions for candidates. Please submit names of able individuals, along with full information about their qualifications, to the Chairman of the Committee on Nominations and Elections, Mr. Eugene Ferguson, Newton High School, Newtonville 60, Massachusetts.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

THIRTY-NINTH ANNUAL MEETING

April 5–8, 1961
Conrad Hilton Hotel, Chicago, Illinois
Robert Sisler, Morton High School West, 2400 Home Avenue, Berwyn, Illinois

JOINT MEETING WITH NEA

June 28, 1961
Atlantic City, New Jersey
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTY-FIRST SUMMER MEETING

August 21–23, 1961
University of Toronto, Toronto, Canada
Father John C. Egsgard, C.S.B., St. Michael's College School, 1515 Bathurst Street, Toronto 10, Canada

Other professional dates

The Greater Cleveland Council of Teachers of Mathematics

February 16, 1961
Roehm Junior High School, Berea, Ohio
Bessie Kisner, Strongsville High School, Strongsville, Ohio

Men's Mathematics Club of Chicago and Metropolitan Area

February 17, 1961
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

Thirty-eighth Annual Joint Meeting of the Louisiana-Mississippi Section of the MAA and the Louisiana-Mississippi Branch of the NCTM

February 17–18, 1961
Buena Vista Hotel, Biloxi, Mississippi
Robert C. Brown, Box 116, Southeastern Louisiana College, Hammond, Louisiana

Mathematics Section, Maryland State Teachers Association

March 18, 1961
Stephens Hall, Towson State Teachers College, Towson, Maryland
W. Edwin Freeny, 507 Milford Mill Road, Baltimore 8, Maryland

Chicago Elementary Teachers Mathematics Club

March 20, 1961
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Mildred C. Rogers, Warren Elementary School, 9210 South Chappel Avenue, Chicago 17, Illinois

The Greater Cleveland Council of Teachers of Mathematics

March 21, 1961
South Euclid School, Cleveland, Ohio
Bessie Kisner, Strongsville High School, Strongsville, Ohio

Illinois Council of Teachers of Mathematics

March 25, 1961 Monticello College, Godfrey, Illinois
April 1, 1961 Southern Illinois University, Carbondale, Illinois
April 14, 1961 Eastern Illinois University, Charleston, Illinois
April 15, 1961 Western Illinois University, Macomb, Illinois
April 22, 1961 Illinois State Normal University, Normal, Illinois
April 29, 1961 Sterling Township High School, Sterling, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

Men's Mathematics Club of Chicago and Metropolitan Area

April 21, 1961
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

Pennsylvania Council of Teachers of Mathematics

April 29, 1961
Clarion State College, Clarion, Pennsylvania
Earl F. Myers, 322 13th Avenue, New Brighton, Pennsylvania

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